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THE DEVELOPMENT OF A DISTRIBUTION PROCEDURE FOR THE ANALYSIS OF CONTINUOUS RECTANGULAR PLATES

By
A. ANG

A Report
To
THE REINFORCED CONCRETE RESEARCH COUNCIL
CORPS OF ENGINEERS, U.S. ARMY
PUBLIC BUILDING SERVICE, GENERAL SERVICES ADMINISTRATION
DIRECTORATE OF CIVIL ENGINEERING, U.S. AIR FORCE
and
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date May 1959

Page 45: Second column, Point 2 should read 0.61526100

Page 63: Should have arrow from [Set (MV) = 1] to [Clear counters (C) and (C1)].

Page 68: Last equation should read,

$$V = (2r - 1)vn + (s - 1)n + 1 + 2(v + 1)hm$$

University of Illinois
Urbana, Illinois

THE DEVELOPMENT OF A DISTRIBUTION PROCEDURE FOR THE ANALYSIS OF CONTINUOUS RECTANGULAR PLATES

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A. Ang

A Report on a Research Project

Sponsored By

THE REINFORCED CONCRETE RESEARCH COUNCIL

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In Cooperation With

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I. INTRODUCTION

1. Object and Scope of Investigation

The study presented here is concerned with the development of a numerical procedure for the analysis of rectangular plates continuous in two directions over flexurally and torsionally stiff beams and rigid columns of rectangular cross-sections. A reinforced concrete multiple-panel floor slab would be typical of such a structure. However, the analysis does not deal with such problems as inelastic action, creep, and moment redistribution, which are inherent in a reinforced concrete slab. The procedure is such that it is perhaps most useful in research application and certainly is not intended for direct use as a routine design procedure. The application of the method, however, is not limited to the analysis of reinforced concrete floor slabs. It can be extended to cover other fields where continuous plates are used as structural elements.

The method of analysis has been developed to consider some of the questions regarding a continuous rectangular plate that have remained unanswered, such as:

(a) continuity in two directions of a plate over beams and columns or columns only,

(b) the effects of both deflections and twist of the supporting beams,

(c) the effects of column sizes,

(d) non-identical loadings in the different panels,

(e) combinations of panels of different shapes,

(f) combinations of panels of different thicknesses.

The method has been applied to the analysis of only a few illustrative problems; an exhaustive study of the above variables is beyond the scope of the present study.

The numerical method of analysis is an extension of the use of finite difference equations, based on the concept of Newmark's plate analog, to the analysis of continuous plates. This involves the determination of the boundary deformations of every individual panel in a structure utilizing a distribution procedure, the essential features of which are based on Cross's moment distribution procedure for beams and frames.[2*]

The method has been coded for completely automatic computation on the ILLIAC[17] (the University of Illinois Digital Computer). The use of these codes on other computers will involve only translation of the codes. However, the amount of calculation is so enormous that the solution of any problem is contingent on the availability of a high-speed digital computer.

2. Historical Review

The analysis of plates has occupied the attention of many researchers, both classical elasticians and those working with approximate solutions. Most of this early work has dealt with isolated panels having different boundary conditions.

More recently, several authors have contributed to the subject of plates continuous over several supports. Marcus[7] and Galerkin[4] both gave solutions to several cases of plates continuous over rigid beams. Jensen[5] presented formulas for a number of cases of rectangular plates supported on two or three flexible beams. Timoshenko[13] gave a formula for the case of a typical interior panel of a plate supported on point columns. Westergaard[15] presented solutions of the same problem by modifying solutions of finite difference equations to account for different column diameters. Sutherland,[12] et. al., treated a problem with point columns and beams of flexural stiffness ranging from 0 to ∞ by the Ritz energy method using the S-functions. Newmark[9] developed a moment

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* Numbers refer to entries in the Bibliography.

and reaction distribution procedure for plates with two opposite edges simply-supported and continuous over flexible or rigid beams in the other direction, with the distribution factors determined by a Fourier series expansion. Siess[11] developed a moment distribution procedure for obtaining the total moments across any section for uniformly loaded plates continuous in two directions over non-deflecting beams. Engelbreth[3] also developed a distribution procedure similar to Siess's. Bittner[1] presented an algebraic method for the analysis of plates continuous over non-deflecting supports by expressing the rotations in terms of all the edge moments and satisfying continuity of resultant slopes between adjoining panels. Maugh and Pan[8] presented a similar method by assuming sine-wave distribution of the edge moments and continuity satisfied only at the middle of each edge.

3. Outline of Thesis

A brief outline of each chapter is presented below:

Chapter II contains a review of the fundamental equations of the ordinary theory of flexure of plates and the derivation of the difference equations directly from Newmark's plate analog.[10]

Chapter III contains an explanation of the basis of the method of analysis and a detailed description of the development and use of the distribution procedure and the method of obtaining the distribution factors.

Chapter IV includes comments regarding the accuracy of solutions by the given method and comparisons with available solutions by other methods.

Chapter V contains several illustrative problems giving the deflections and moments for each problem.

Chapter VI is a summary.

Chapter VII is a bibliography.

Appendix A contains the flow diagrams of each program.

Appendix B contains brief descriptions of the programs and the detailed write-ups of each program and its important sub-routines.

4. Basic Assumptions

The method of analysis presented here is based on the ordinary theory of flexure of plates. The assumptions involved in the analysis may be stated as follows:

(a) The plate is of constant thickness and is subjected to loads normal to its middle plane only.

(b) The plate is of homogeneous, elastic, and isotropic material.

(c) The stresses acting on any cross-section have no resultant force in the direction of the plane of the plate.

(d) The deformations of the plate are the deformations of its middle plane.

(e) Plane vertical sections of the undeformed plate remain plane under a deformation.

In addition, the beams are assumed to have their neutral axes in the same level as the middle plane of the plate. In order to account for the T-beam effect, a modified stiffness of the beams may be used. The forces and moments acting on a beam are assumed to be distributed along a line and not over a finite width. The columns are of rectangular sections and are assumed to be infinitely stiff.

5. Acknowledgment

The investigation of multiple-panel reinforced concrete floor slabs is being conducted in the Structural Research Laboratory of the Department of Civil Engineering at the University of Illinois in cooperation with the Reinforced Concrete Research Council of the Engineering Foundation; the Office of the Chief of Engineers, Corps of Engineers, U.S. Army; Public Building Service, General Services Administration; and the Directorate of Construction, Headquarters, U. S. Air Force. The project was formerly divided into an experimental phase and an analytical phase. The first portion of the work in this thesis was done under the analytical phase of this project.

The latter portion of this investigation was done under the sponsorship of the National Science Foundation.

The above-mentioned projects are under the direction of Dr. C. P. Siess, Professor of Civil Engineering, and this thesis is also written under his direction.

The author wishes to express his thanks to Dr. C. P. Siess for his direction and encouragement, and to Dr. J. E. Duberg, Professor of Civil Engineering, for his constructive criticisms and suggestions, regarding this study. Mr. W. Hsiong, Research Assistant in Civil Engineering, prepared some of the figures and curves and his help is appreciated. Profound thanks are due to the staff of the ILLIAC for allowing machine time for solving the problems and code-checking of the programs.

The work in this thesis grew out of a discussion with Dr. N. M. Newmark, Professor of Civil Engineering and Head of the Department of Civil Engineering of the University of Illinois. His advice and suggestions, that lead to the development of the procedure, are gratefully acknowledged.

6. Notations

The following notations have been adopted for use in this thesis:

x, y, z = rectangular reference coordinates; x and y are also used to designate the columns and rows of a matrix

ϵ = strain*

γ = shearing strain*

σ = stress*

τ = shearing stress*

u, v = deformations in the x- and y-directions, respectively

w = final deflection of a plate or beam, positive downward

w_c or $w_c(x,y)$ = deflection at point (x,y) due to boundary deformations

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* Positive sense of these strains and stresses are the same as those used in Ref. 14.

w_p = deflection at point (x,y) due to a given loading with special boundary conditions.

$z_{1i}(x,y)$ = deflection at point (x,y) due to a rotation φ_i at joint i only

$z_{2i}(x,y)$ = deflection at point (x,y) due to a deflection δ_i at joint i only

δ_i, δ_{ii}, δ_{ji}, $\bar{\delta}_j$, Δ_j = edge deflections, defined in Sections 10 and 13

φ_i, φ_{ii}, φ_{ji}, $\bar{\varphi}_j$, θ_j = edge rotations, defined in Sections 10 and 13

$\alpha_i(x,y)$, $\beta_i(x,y)$ = deflection coefficients, defined in Section 10

h_{ji}, f_{ji}, g_{ji}, g'_{ji} = moment and reaction coefficients, defined in Section 11

Ω_{ji}, ψ_{ji} = rotation and deflection coefficients respectively, defined in Section 12

K_i, T_i, k_{ji}, t_{ji}, q_{ji}, q'_{ji} = distribution factors, defined in Section 13

m_{ji}, r_{ji}, $\bar{m}_j$, $\bar{r}_j$ = moments and reactions at j, defined in Sections 13 and 14

M = bending or twisting moments

V = shear forces

R = reactions

p = distributed load per unit of area

Q = total load acting at a node point

λ, λ_x, λ_y = distances between node points of a network

$\xi = \lambda_y/\lambda_x$, a ratio

$I = \frac{h^3}{12}$ = moment of inertia per unit of width of plate in a particular panel

E = modulus of elasticity of the material of the plate in a particular panel

μ = Poisson's ratio, taken as zero in the illustrative problems

$N = \frac{Eh^3}{12(1 - \mu^2)}$ = a measure of the stiffness of the plate in a particular panel

E_b = modulus of elasticity of the material in a beam

I_b = moment of inertia of the cross-section of a beam

$G = \frac{E}{2(1 + \mu)}$ = shear modulus of elasticity of the material in a beam

$C = \frac{b^3 d}{3} f_1$ = a measure of the torsional rigidity of a beam (See Ref. 11)

$H = E_b I_b / N\lambda$ = a ratio of beam flexural stiffness to plate stiffness

$J = GC/N\lambda$ = a ratio of beam torsional stiffness to plate stiffness

m,n = number of joints in an edge undergoing deformations

a,b = span lengths of a plate

c = cycle number of a calculation

A "joint" is defined as an elastic hinge or node point on an edge of a panel or common to two panels. An edge in the east-west direction is called an "E-W edge" and an edge in the north-south direction is called an "N-S edge".

II. DERIVATION OF EQUATIONS

7. Review of Fundamental Equations

Based on the assumptions (a) through (e) listed in Section (4), the equations of the ordinary theory of flexure of plates can be found in a number of places.

The differential equation governing the deflection, w, of the plane of the plate is:

$$\nabla^2 \cdot \nabla^2 w = p/N \tag{1}$$

where: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator for two variables.

The bending and twisting moments are related to the deflection by the following equations:

$$\left.\begin{aligned} M_x &= -N\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -N\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -N\,(1-\mu)\,\frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \tag{2}$$

The shears are:

$$\left.\begin{aligned} V_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -N\,\frac{\partial}{\partial x}\,(\nabla^2 w) \\ V_y &= \frac{\partial M_y}{\partial_y} + \frac{\partial M_{xy}}{\partial x} = -N\,\frac{\partial}{\partial y}\,(\nabla^2 w) \end{aligned}\right\} \tag{3}$$

The reactions are:

$$\left.\begin{aligned} R_x &= V_x + \frac{\partial M_{xy}}{\partial y} = -N\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x \partial y^2}\right] \\ R_y &= V_y + \frac{\partial M_{xy}}{\partial x} = -N\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right] \end{aligned}\right\} \tag{4}$$

At each corner formed by the intersection of two non-deflecting faces normal to the x- and y-axes, there are concentrated corner reactions, R_c, the magnitude of which are:

$$R_c = 2M_{xy} \tag{5}$$

The ordinary theory of flexure of plates neglects the effects of stresses normal to the plane of the plate and also resultant stresses in the plane of the plate. This theory, therefore, is applicable only to certain types of plates, commonly referred to as "medium thick plates", and also does not apply to the neighborhood of a concentrated load.

8. Finite Difference Equations

Problems associated with the mechanics of continua involve ordinary or partial differential equations. These equations result from taking infinitesimally small differentials in setting up a problem. The solutions to these differential equations are, in general, continuous functions.

If in place of the differentials, a problem is approximated by taking small finite lengths, the difference equations corresponding to the differential equations are obtained. The solutions by difference equations are discrete quantities and theoretically these solutions approach that of the differential equations as the small finite lengths approach zero. Therefore, the degree of approximation will, in general, be improved by taking smaller finite lengths.

The use of difference equations in place of the corresponding differential equations reduces a problem of infinite degree of indeterminateness to one of a finite degree.

The difficulty concerning concentrated loads is circumvented when using finite differences, since a concentrated load is assumed to be uniformly distributed over an area $\lambda_x\lambda_y$, or $p = P/\lambda_x\lambda_y$ at the loaded points, where P is the concentrated load.

(a.) Concept of Newmark's Plate Analog.

The solution to a problem of a plate continuous over several supports, as given by the method presented here, is based on the interconnection of piecewise solutions of individual panels. This interconnection of the piecewise solutions involves satisfying certain physical requirements. The use of difference equation formulas derived from the differential equation, Eq. (1), are two stages removed from the physical phenomenon itself.[6] This difficulty can be rectified by using a physical model of the plate from which the same difference equations can be derived directly. The physical model used in this dissertation was developed by Dr. N. M. Newmark and is referred to here as "Newmark's plate analog".[10] A brief review of this analog, its physical properties and the difference equations derived directly from the analog, is presented below.

Newmark's plate analog is composed of rigid bars connecting elastic hinges with torsion springs attached to adjacent parallel bars, Fig. 1a, and is assumed to have the following properties:

(i) The bars are weightless and undeformable.

(ii) All the strains due to direct stresses occur only at the elastic hinges, where the mass is also concentrated. These strains are the average strains over a width λ_x or λ_y. The resultant of these average direct stresses are bending moments acting at the hinges and at the ends of each bar.

(iii) The shearing strains in a block enclosed by four bars are average shearing strains for the block. The resultant of the average horizontal shearing stresses are twisting moments concentrated in the torsion springs.

(b.) Difference Operators.

The difference equations governing the deflections of the plate can be derived directly from this analog as follows:

Referring to Fig. 2a, equilibrium of moments for bar ao gives:

$$(V_{aox} \cdot \lambda_y)\,\lambda_x = (M_{ax} - M_{ox})\,\lambda_y + (M_{Axy} - M_{Bxy})\,\lambda_x$$

The forces and moments in this equation are average quantities over a width λ_x or λ_y. This can be written also as:

$$V_{aox} = \frac{M_{ax} - M_{ox}}{\lambda_x} + \frac{M_{Axy} - M_{Bxy}}{\lambda_y} \tag{6}$$

which is equivalent to the differential equation

$$V_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

Similarly, for bars oc, eo, and og;

$$\left.\begin{aligned} V_{ocx} &= \frac{M_{ox} - M_{cx}}{\lambda_x} + \frac{M_{Dxy} - M_{Cxy}}{\lambda_y} \\ V_{eoy} &= \frac{M_{ey} - M_{oy}}{\lambda_y} + \frac{M_{Axy} - M_{Dxy}}{\lambda_x} \\ V_{ogy} &= \frac{M_{oy} - M_{gy}}{\lambda_y} + \frac{M_{Bxy} - M_{Cxy}}{\lambda_x} \end{aligned}\right\} \tag{6}$$

Equilibrium of vertical forces of the hinge o gives:

$$V_{aox}\,\lambda_y - V_{ocx}\,\lambda_y + V_{eoy}\,\lambda_x - V_{ogy}\,\lambda_x = Q$$

Using $\overline{V}_{aox}$, $\overline{V}_{ocx}$, $\overline{V}_{eoy}$, and $\overline{V}_{ogy}$ as the total forces acting, this equation is,

$$\overline{V}_{aox} - \overline{V}_{ocx} + \overline{V}_{eoy} - \overline{V}_{ogy} = Q \tag{7}$$

The average strains at o and the average shearing strains in A are:

$$\epsilon_{ox} = \frac{u_{ao} - u_{oc}}{\lambda_x}$$

$$\epsilon_{oy} = \frac{v_{eo} - v_{og}}{\lambda_y}$$

$$\gamma_{Axy} = \frac{u_{ie} - u_{ao}}{\lambda_y} + \frac{v_{ia} - v_{eo}}{\lambda_x}$$

and using Hooke's emperical relationship (neglecting σ_z):

$$\epsilon_{ox} = \frac{1}{E} (\sigma_{ox} - \mu\sigma_{oy})$$

$$\epsilon_{oy} = \frac{1}{E} (\sigma_{oy} - \mu\sigma_{ox})$$

$$\gamma_{Axy} = \frac{1}{G} \tau_{Axy}$$

yields the following:

$$\sigma_{ox} = \frac{E}{1 - \mu^2} (\epsilon_{ox} + \mu\epsilon_{oy})$$

$$\sigma_{oy} = \frac{E}{1 - \mu^2} (\epsilon_{oy} + \mu\epsilon_{ox})$$

$$\tau_{Axy} = G \cdot \gamma_{Axy} = \frac{E}{2(1 + \mu)} \gamma_{Axy}$$

The compatability condition gives the following relationship for the three components of deformation:

Referring to Fig. 2b, since bar ao is undeformable, the deformation u_{ao} is the deformation at the hinges o or a due to w_o and w_a, which is

$$u_{ao} = z(\frac{w_a - w_o}{\lambda_x})$$

where z is the vertical distance from the middle plane of the plate. Likewise;

$$u_{oc} = z(\frac{w_o - w_c}{\lambda_x})$$

$$u_{ie} = z(\frac{w_i - w_e}{\lambda_x})$$

and

$$v_{eo} = z(\frac{w_e - w_o}{\lambda_y})$$

$$v_{og} = z\left(\frac{w_o - w_g}{\lambda_y}\right)$$

$$v_{ia} = z\left(\frac{w_i - w_a}{\lambda_y}\right) .$$

From which the average strains are:

$$\epsilon_{ox} = \frac{z}{\lambda_x^2}(w_a - 2w_o + w_c)$$

$$\epsilon_{oy} = \frac{z}{\lambda_y^2}(w_e - 2w_o + w_g)$$

$$\gamma_{Axy} = \frac{z}{\lambda_x\lambda_y}\left\{[(w_i - w_e) - (w_a - w_o)] + [(w_i - w_a) - (w_e - w_o)]\right\}$$

$$= \frac{2z}{\lambda_x\lambda_y}[(w_i - w_e) - (w_a - w_o)]$$

and the average stresses are:

$$\sigma_{ox} = \frac{E}{(1 - \mu^2)} \cdot z\left[\frac{(w_a - 2w_o + w_c)}{\lambda_x^2} + \mu\frac{(w_e - 2w_o + w_g)}{\lambda_y^2}\right]$$

$$\sigma_{oy} = \frac{E}{(1 - \mu^2)} \cdot z\left[\frac{(w_e - 2w_o + w_g)}{\lambda_y^2} + \mu\frac{(w_a - 2w_o + w_c)}{\lambda_x^2}\right]$$

$$\tau_{Axy} = \frac{E}{(1 + \mu)} \cdot z\left[\frac{(w_i - w_e) - (w_a - w_o)}{\lambda_x\lambda_y}\right]$$

The average bending and twisting moments are therefore:

$$\left.\begin{aligned}
M_{ox} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ox} \cdot z\,dz = -N\left[\frac{(w_a - 2w_o + w_c)}{\lambda_x^2} + \mu\,\frac{(w_e - 2w_o + w_g)}{\lambda_y^2}\right] \\
M_{oy} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{oy} \cdot z\,dz = -N\left[\frac{(w_e - 2w_o + w_g)}{\lambda_y^2} + \mu\,\frac{(w_a - 2w_o + w_c)}{\lambda_x^2}\right] \\
M_{Axy} &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{Axy} \cdot z\,dz = -N(1 - \mu)\left[\frac{(w_i - w_e) - (w_a - w_o)}{\lambda_x \lambda_y}\right]
\end{aligned}\right\} \quad (8)$$

where $N = \dfrac{Eh^3}{12(1 - \mu^2)}$.

Substituting Eq. (8) and its companion equations into Eq. (6) and the result substituted into Eq. (7) gives the following equation:

$$\begin{aligned}
&[\,6 + 8\xi^2 + 6\xi^4]\;w_o - 4(1 + \xi^2)\,[(w_e + w_g) + \xi^2(w_a + w_c)] \\
&+ (w_f + w_h) + \xi^4(w_b + w_d) + 2\xi^2(w_i + w_j + w_m + w_n) = \frac{\xi^3\lambda_x^2 Q}{N}
\end{aligned} \qquad (9)$$

where $\xi = \lambda_y/\lambda_x$.

(i) <u>General Interior Point</u>:

Equation (9) is the difference equation for a rectangular net corresponding to the differential equation $\nabla^2\,\nabla^2\,w = p/N$ for an interior point. In operator form, this is:

$$\left\{\begin{array}{ccccc} & & 1 & & \\ & 2\xi^2 & -4(1+\xi^2) & 2\xi^2 & \\ \xi^4 & -4\xi^2(1+\xi^2) & (6+8\xi^2+6\xi^4) & -4\xi^2(1+\xi^2) & \xi^4 \\ & 2\xi^2 & -4(1+\xi^2) & 2\xi^2 & \\ & & 1 & & \end{array}\right\} w = \frac{\xi^3\lambda_x^2 Q}{N} \qquad (0\text{-}1)$$

(ii) <u>Point on an Edge Beam</u>:

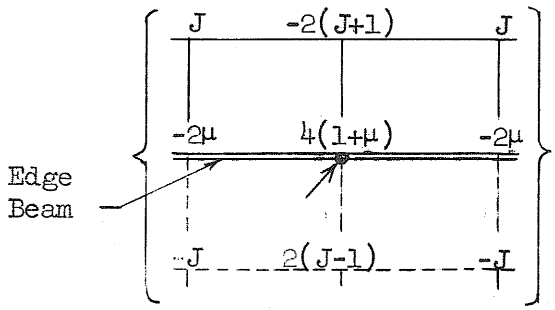
Newmark's plate analog can be extended to include the presence of a beam acting with the plate, assuming that the neutral axis of the beam is in the same level as the middle plane of the plate. The beam is represented by weightless, rigid bars connecting elastic joints, with torsion springs wrapped around the bars to resist torsion, Fig. 1b.

The only case of interest here is when the beam is on an edge of the plate. The boundary conditions to be satisfied between the plate and the beam are:

(a) The bending moments acting on the edge of the plate is equal and opposite to the torsional moment acting on the beam;

(b) The reaction on the edge of the plate is equal and opposite to the load applied on the beam.

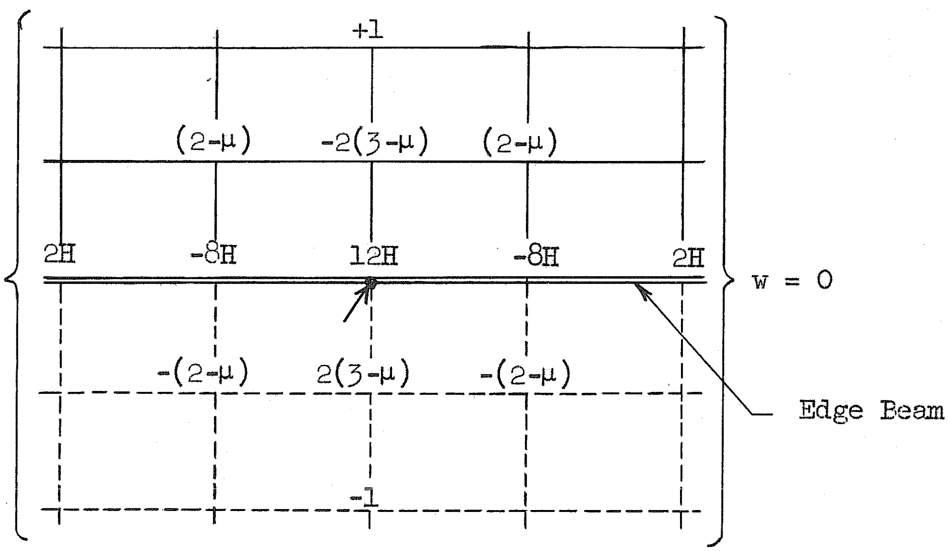
In operator form, boundary condition (a), for $\xi = 1$, is:



$w = 0$ (BC-1)

where $J = \frac{GC}{N\lambda}$.

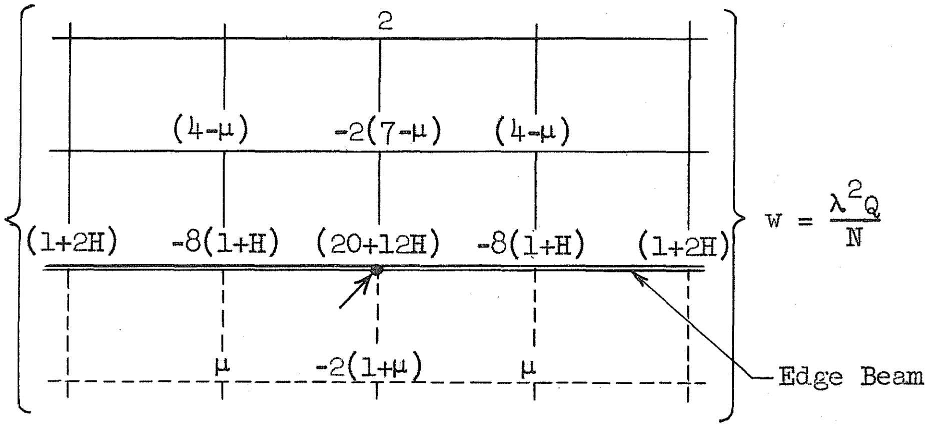
In operator form, boundary condition (b), for $\xi = 1$, is:



$w = 0$ (BC-2)

where $H = \frac{E_b I_b}{N\lambda}$.

If the operator (O-1) is applied to a point on an edge beam there obviously will be four points outside the edge which will require four side conditions to determine. Applying (BC-2) to the same point on the edge beam as (O-1), the point farthest from the edge can be eliminated and the following operator is obtained:



$$w = \frac{\lambda^2 Q}{N} \tag{0-2}$$

The fictitious points immediately adjacent to the edge will not be eliminated. It is more convenient to apply (BC-1) directly to each point on the edge beam, thus providing the same number of extra equations as there are fictitious points adjacent to the edge. The solution, therefore, will involve the deflections of these fictitious points in addition to the interior and edge points. Physically, the deflections of these fictitious points are necessary to determine the twist in the edge beam.

III. METHOD OF ANALYSIS

9. Basis of Method of Analysis

The complete solution, in terms of the deflections, w, of any one panel within a general continuous plate may be said to be composed of the solution of the panel with arbitrarily specified boundary conditions subjected to the given loading plus the solution due to the boundary deformations necessary to satisfy the conditions of equilibrium and continuity with the adjoining panels. The complete solution may be expressed as:

$$w = w_p + w_c$$

where;

w_p = the solution satisfying the plate equation $\nabla^2\nabla^2 w = p/N$, and certain arbitrarily prescribed boundary conditions for the panel,

w_c = the solution satisfying the homogeneous equation $\nabla^2\nabla^2 w = 0$, and the boundary deformations of the panel which are necessary to maintain equilibrium and continuity with the other panels of a structure.

This may be made clearer by showing the one-dimensional analog, which is the solution to the problem of a continuous beam, such as the one shown in Fig. 3a. The differential equation governing the deflections of a beam in bending is:

$$\frac{d^4w}{dx^4} = \frac{p}{EI} \tag{10}$$

The total solution, w, for span II may be divided into:

$$w = w_p + w_c$$

where:

w_p = the solution satisfying Eq. (10) with both ends of span II fixed, which is Fig. 3b.

w_c = the solution satisfying $\dfrac{d^4w}{dx^4} = 0$

and the boundary deformations

$$w_o = 0;\ w_L = 0$$

$$\left.\frac{dw}{dx}\right]_{x=0} = \theta_1\ ;\ \left.\frac{dw}{dx}\right]_{x=L} = \theta_2$$

where θ_1 and θ_2 are the rotations at the ends of span II necessary for equilibrium and continuity with spans I and III.

The main problem, therefore, in the analysis of a continuous plate by this method is the determination of the boundary deformations of each panel and the determination of w_c. The partial solution w_p is determined by direct application of finite difference equations to a panel having zero slopes and zero deflections at all the edges as its boundary conditions. This will involve 1/4 of the panel for rectangular panels and 1/8 of the panel for square panels because of symmetry.

10. <u>Relation Between w_c and the Boundary Deformations</u>

In Fig. 4a, let joint i be subjected to a rotation φ_i and prevented from deflection. All other joints, including those on edge A, are held against both rotation and deflection. The deflections of the interior points due to φ_i, denoted as z_{1i} (x,y) are:

$$z_{1i}(x,y) = \alpha_i(x,y) \cdot \varphi_i$$

Similarly, if joint i is subjected to a deflection δ_i, Fig. 4b, and prevented from rotation, and all other joints are held against both rotation and deflection, the deflections of the interior points, $z_{2i}(x,y)$, are:

$$z_{2i}(x,y) = \beta_i(x,y) \cdot \delta_i$$

Therefore, the deflections of the interior points, $w_c(x,y)$, due to all the edge rotations, φ_i, and edge deflections, δ_i, at all the joints on the entire perimeter of a panel, are:

$$w_c(x,y) = \sum_{i=1}^{2(m+n)} \left[z_{1i}(x,y) + z_{2i}(x,y)\right]$$

$$= \sum_{i=1}^{2(m+n)} \left[\alpha_i(x,y)\ \varphi_i + \beta_i(x,y)\ \delta_i\right]$$

where:

m, n = number of joints in an E-W edge and N-S edge, respectively.

$\alpha_i(x,y)$ = deflection coefficient for an interior point (x,y) due to φ_i.

$\beta_i(x,y)$ = deflection coefficient for an interior point (x,y) due to δ_i.

The determination of $\alpha_i(x,y)$ and $\beta_i(x,y)$ involves the solution of a set of simultaneous equations; the number of equations for each φ_i or δ_i, being equal to the number of interior points in a panel, where, for rectangular panels,

$$i = 1, 2, \ldots\ldots, \left(\frac{m}{2} + \frac{n}{2}\right) \qquad \text{for } m, n \text{ even}$$

or

$$i = 1, 2, \ldots\ldots, \left(\frac{m+1}{2} + \frac{n+1}{2}\right) \qquad \text{for } m, n \text{ odd}$$

and for square panels,

$$i = 1, 2, \ldots\ldots, \frac{m}{2} \qquad \text{for } m \text{ even}$$

or

$$i = 1, 2, \ldots\ldots, \frac{m+1}{2} \qquad \text{for } m \text{ odd.}$$

The coefficients $\alpha_i(x,y)$ and $\beta_i(x,y)$ are independent of the beam stiffnesses but are functions of the aspect ratio of a panel and the column sizes.

11. Determination of Boundary Deformations:

(a.) Slope-Deflection Equations

The unknown boundary rotations and deflections can be determined by setting up equilibrium equations as follows:

Referring to Figs. 4a and 4b let

h_{ji} = the moment at j due to $\varphi_i = 1$,

g_{ji} = the reaction at j due to $\varphi_i = 1$,

f_{ji} = the reaction at j due to $\delta_i = 1$,

g'_{ji} = the moment at j due to $\delta_i = 1$.

The total moment at j due to the boundary deformations φ_i and δ_i is therefore:

$$M_j = \sum_{i=1}^{2(m+n)} h_{ji} \cdot \varphi_i + \sum_{i=1}^{2(m+n)} g'_{ji} \cdot \delta_i$$

and the corresponding total reaction at j is:

$$R_j = \sum_{i=1}^{2(m+n)} g_{ji} \cdot \varphi_i + \sum_{i=1}^{2(m+n)} f_{ji} \cdot \delta_i$$

In matrix notation, these moments for an E-W edge of a panel are:

$$\left\| \begin{matrix} M_1 \\ M_2 \\ \vdots \\ \vdots \\ M_j \\ \vdots \\ \vdots \\ \vdots \\ M_m \end{matrix} \right\| = \left\| \begin{matrix} h_{11}, & h_{12}, & \cdots & h_{1i}, & \cdots & h_{1\bar{n}}; & g'_{11}, & g'_{12}, & \cdots & g'_{1i}, & \cdots & g'_{1\bar{n}} \\ h_{21}, & h_{22}, & \cdots & h_{2i}, & \cdots & h_{2\bar{n}}; & g'_{21}, & g'_{22}, & \cdots & g'_{2i}, & \cdots & g'_{2\bar{n}} \\ \cdots & & & & & & & & & & & \cdots \\ \cdots & & & & & & & & & & & \cdots \\ h_{j1}, & h_{j2}, & \cdots & h_{ji}, & \cdots & h_{j\bar{n}}; & g'_{j1}, & g'_{j2}, & \cdots & g'_{ji}, & \cdots & g'_{j\bar{n}} \\ \cdots & & & & & & & & & & & \cdots \\ \cdots & & & & & & & & & & & \cdots \\ \cdots & & & & & & & & & & & \cdots \\ h_{m1}, & h_{m2}, & \cdots & h_{mi}, & \cdots & h_{m\bar{n}}; & g'_{m1}, & g'_{m2}, & \cdots & g'_{mi}, & \cdots & g'_{m\bar{n}} \end{matrix} \right\| \cdot \left\| \begin{matrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_i \\ \vdots \\ \varphi_{\bar{n}} \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_i \\ \vdots \\ \delta_{\bar{n}} \end{matrix} \right\|$$

where $\bar{n} = 2(m+n)$

or

$$\underline{M} = \underline{K} \cdot \underline{\theta}^*$$

where $\underline{K}$ = the moment coefficient matrix

*The underlined symbols represent matrices.

The corresponding reactions are:

$$\left\| \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ \vdots \\ R_j \\ \vdots \\ \vdots \\ R_m \end{array} \right\| = \left\| \begin{array}{l} g_{11},\ g_{12},\ \cdots\ g_{1i},\ \cdots\ g_{1\bar{n}}\ ;\ f_{11},\ f_{12},\ \cdots\ f_{1i},\ \cdots\ f_{1\bar{n}} \\ g_{21},\ g_{22},\ \cdots\ g_{2i},\ \cdots\ g_{2\bar{n}}\ ;\ f_{21},\ f_{22},\ \cdots\ f_{2i},\ \cdots\ f_{2\bar{n}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_{j1},\ g_{j2},\ \cdots\ g_{ji},\ \cdots\ g_{j\bar{n}}\ ;\ f_{j1},\ f_{j2},\ \cdots\ f_{ji},\ \cdots\ f_{j\bar{n}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_{m1},\ g_{m2},\ \cdots\ g_{mi},\ \cdots\ g_{m\bar{n}}\ ;\ f_{m1},\ f_{m2},\ \cdots\ f_{mi},\ \cdots\ f_{m\bar{n}} \end{array} \right\| \cdot \left\| \begin{array}{c} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_i \\ \vdots \\ \varphi_{\bar{n}} \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_i \\ \vdots \\ \delta_{\bar{n}} \end{array} \right\|$$

or
$$\underline{R} = \underline{S} \cdot \underline{\theta}$$

where $\underline{S}$ = the reaction coefficient matrix.

These moments and reactions are those for an edge of a particular panel within a structure. The condition of equilibrium at every joint in a given structure yields the following equations in matrix form:

For a given edge N:

$$\underline{\theta_N} \cdot \underline{K_R} + \underline{\theta_N} \cdot \underline{K_L} + \underline{U_N} = 0$$

or

$$\underline{\theta_N}(\underline{K_R} + \underline{K_L}) + \underline{U_N} = 0 \tag{11}$$

also

$$\underline{\theta_N} \cdot \underline{S_R} + \underline{\theta_N} \cdot \underline{S_L} + \underline{W_N} = 0$$

or

$$\underline{\theta_N}(\underline{S_R} + \underline{S_L}) + \underline{W_N} = 0 \tag{12}$$

where the subscripts R and L indicate the panels to the right and left, respectively, of the edge, and

$\underline{U_N}$, $\underline{W_N}$ = the load factor matrix.

The solution of Eqs. (11) and (12) determines the unknown boundary rotations and deflections of every panel. The number of equations, which is equal to the number of unknowns, is twice the number of edge joints.

(b.) Distribution Procedure

An alternative to solving Eqs. (11) and (12) directly for determining the boundary rotations and deflections is a distribution procedure which, in principle, is related to Eqs. (11) and (12) as Cross's moment distribution procedure is to the slope-deflection equations for beams and frames. The essential features of the distribution procedure are based on Cross's moment distribution procedure for beams and frames.

The relaxation of an unbalanced force at a given joint, however, does not confine a deformation to that joint only, but rather the other joints in the same edge are allowed to deform in a prescribed fashion. The elements of the matrices in Eqs. (11) and (12) are therefore not exactly related to the distribution factors used here as the terms in the slope-deflection equations for frames are related to the distribution factors used in Cross's moment distribution procedure for frames.

12. Determination of w_c

If θ_j and Δ_j are respectively the boundary rotations and boundary deflections of a panel within a continuous structure, which satisfy Eqs. (11) and (12), then the deflections of the interior points, $w_c(x,y)$ are determined by the relationships given in Section 10, which are:

$$w_c(x,y) = \sum_{j=1}^{2(m+n)} [\alpha_j(x,y) \cdot \theta_j + \beta_j(x,y) \cdot \Delta_j]$$

13. Distribution Factors for a Panel

(a) The Distribution Factors

Consider the plate shown in Fig. 5a. Three of the edges, B, C and D, are held against both rotation and deflection. The fourth edge, A, is free to rotate, but prevented from deflecting.

Let joint i be subjected to a specified rotation φ_{ii}. The other joints on this edge will also rotate an amount

$$\varphi_{ji} = \Omega_{ji} \cdot \varphi_{ii} .$$

Ω_{ji} is herein called the "edge rotation factor".

The moment required at i to produce the rotation φ_{ii} is

$$M_i = K_i \cdot \varphi_{ii}$$

or

$$K_i = \frac{M_i}{\varphi_{ii}}$$

which is termed the "flexural stiffness" of the plate at joint i. This is defined as the moment required at i to produce a unit rotation at joint i.

Due to this rotation, φ_{ii}, reactions will be induced at the other joints of A, which are;

$$r_{ji} = q_{ji} \cdot \varphi_{ii}$$

Since A is free to rotate, the moments at the other joints of A are zero.

Moments and reactions at the edges B, C and D, resulting from φ_{ii} are respectively;

$$m_{ji} = k_{ji} \cdot \varphi_{ii}$$

and

$$r_{ji} = q_{ji} \cdot \varphi_{ii}$$

k_{ji} is called the "flexural carry-over factor" and q_{ji} the "flexure-shear carry-over factor" from i to j.

Now let joint i be subjected to a specified deflection δ_{ii}, Fig. 5b, and the edge A free to deflect but prevented from rotating, with edges B, C and D remaining as before. The deflections on A are:

$$\delta_{ji} = \psi_{ji} \cdot \delta_{ii}$$

where ψ_{ji} is called here the "edge deflection factor".

The vertical force required at i to produce the deflection δ_{ii} is

$$R_i = T_i \cdot \delta_{ii}$$

or

$$T_i = \frac{R_i}{\delta_{ii}} \quad ,$$

which is termed the "shear stiffness" of the plate at joint i. This is defined as the vertical force required at i to produce a unit deflection at joint i.

Due to this deflection, δ_{ii}, the moments at the other joints of A are,

$$m_{ji} = q'_{ji} \cdot \delta_{ii}$$

The reactions at the other joints of A are zero, since this edge is free to deflect.

The moments and reactions at edges B, C and D are respectively;

$$m_{ji} = q'_{ji} \cdot \delta_{ii}$$

and

$$r_{ji} = t_{ji} \cdot \delta_{ii}$$

t_{ji} is called the "shear carry-over factor" and q'_{ji} the "shear-flexural carry-over factor" from i to j.

(b.) Method of Obtaining the Distribution Factors

The abovementioned distribution factors can be determined by direct application of finite differences to Figs. 5a and 5b, with the use of Operator (BC-1) for Fig. 5a, and Operator (O-2) for Fig. 5b, for the points on the edge beam. However, these factors can be determined also from the results of Section 11(a) and are related to the elements of the coefficient matrices of Eqs. (11) and (12) as follows:

If the joints on A, in Fig. 4a, are allowed to rotate only, then let $\varphi_i = \varphi_{ii}$ and the rotations at the other joints on A are determined from the condition that $M_j = 0$ for all joints on A, which is:

$$h_{j1}(\Omega_{1i} \cdot \varphi_i) + h_{j2}(\Omega_{2i} \cdot \varphi_i) + \ldots + h_{jm}(\Omega_{mi} \cdot \varphi_i) = 0$$

or

$$\sum_{x=1}^{m} h_{jx}(\Omega_{xi} \cdot \varphi_i) = 0$$

or

$$\sum_{x=1}^{m} h_{jx} \cdot \Omega_{xi} = 0 \; ; \qquad x = 1, 2, \ldots., i, \ldots., m$$

In matrix notation, these are:

$$\begin{Vmatrix} h_{11}, & h_{12}, & \ldots & h_{1i}, & \ldots & h_{1m} \\ h_{21}, & h_{22}, & \ldots & h_{2i}, & \ldots & h_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{i1}, & h_{i2}, & \ldots & h_{ii}, & \ldots & h_{im} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{m1}, & h_{m2}, & \ldots & h_{mi}, & \ldots & h_{mm} \end{Vmatrix} \cdot \begin{Vmatrix} \Omega_{1i} \\ \Omega_{2i} \\ \cdot \\ \cdot \\ \Omega_{ii} \\ \cdot \\ \Omega_{mi} \end{Vmatrix} = 0 \qquad (13)$$

where $\Omega_{ii} = 1$.

Equation (13) with the row $(h_{i1}, h_{i2}, \ldots. h_{ii}, \ldots. h_{im})$ deleted from the first matrix forms the equations for determining the "edge rotation factors", Ω_{ji}, of all joints on A. The deleted submatrix multiplied by the Ω matrix gives the moment, K_i, required at i to produce $\varphi_{ii} = 1$, or

$$K_i = \sum_{x=1}^{m} h_{ix} \cdot \Omega_{xi}$$

Similarly, if the joints on A, in Fig. 4b are allowed to deflect only, then let $\delta_i = \delta_{ii}$ and the deflections at the other joints on A are determined from the condition that $R_j = 0$ for all joints on A, which in matrix notation, are:

$$\begin{Vmatrix} f_{11}, & f_{12}, & \cdots & f_{1i}, & \cdots & f_{1m} \\ f_{21}, & f_{22}, & \cdots & f_{2i}, & \cdots & f_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{i1}, & f_{i2}, & \cdots & f_{ii}, & \cdots & f_{im} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{m1}, & f_{m2}, & \cdots & f_{mi}, & \cdots & f_{mm} \end{Vmatrix} \cdot \begin{Vmatrix} \psi_{1i} \\ \psi_{2i} \\ \vdots \\ \psi_{ii} \\ \vdots \\ \psi_{mi} \end{Vmatrix} = 0 \tag{14}$$

where $\psi_{ii} = 1$.

Equation (14) with the row $(f_{i1}, f_{i2}, \ldots f_{ii}, \ldots f_{im})$ deleted from the first matrix forms the equations for determining the "edge deflection factors", ψ_{ji}, of all joints on A. The deleted submatrix multiplied by the ψ matrix gives the vertical force, T_i, required at i to produce $\delta_{ii} = 1$, or

$$T_i = \sum_{x=1}^{m} f_{ix} \cdot \psi_{xi}$$

After the Ω_{ji} and ψ_{ji} are found, the other distribution factors are determined as follows:

Due to $\varphi_{ii} = \varphi_i$:

$$m_{ji} = \sum_{x=1}^{m} h_{jx}(\Omega_{xi} \cdot \varphi_i)$$

therefore,

$$k_{ji} = \sum_{x=1}^{m} h_{jx} \cdot \Omega_{xi}$$

except that the moments at A are zero.

Also,

$$r_{ji} = \sum_{x=1}^{m} g_{jx}(\Omega_{xi} \cdot \varphi_{i})$$

therefore,

$$q_{ji} = \sum_{x=1}^{m} g_{jx} \cdot \Omega_{xi}$$

Due to $\delta_{ii} = \delta_{i}$:

$$m_{ji} = \sum_{x=1}^{m} g'_{jx}(\psi_{xi} \cdot \delta_{i})$$

therefore,

$$q'_{ji} = \sum_{x=1}^{m} g'_{jx} \cdot \psi_{xi}$$

Also,

$$r_{ji} = \sum_{x=1}^{m} f_{jx}(\psi_{xi} \cdot \delta_{i})$$

therefore,

$$t_{ji} = \sum_{x=1}^{m} f_{jx} \cdot \psi_{xi}$$

except that the reactions at A are zero.

The values of these distribution constants when obtained may be checked by the reciprocal theorem.

The above equations are for E-W edges. The same equations apply to N-S edges by replacing all the m's by n's wherever m occurs.

These distribution factors are functions of the beam stiffnesses as well as the aspect ratio of a panel and the column sizes. However, a change in the beam stiffnesses will only affect the factors Ω_{ji} and ψ_{ji}, which will only involve the solution of Eqs. (13) and (14), where each equation is comprised of i sets of m or n equations depending on whether i is on an E-W edge or an N-S edge.

14. Details of the Distribution Procedure

The distribution procedure is a method of successively correcting an initial set of forces on the edges of all panels due to a given loading on the structure with certain arbitrarily prescribed boundary conditions. Each successive correction balances one force (moment or reaction) at a time at each joint common to two panels, while maintaining continuity at all joints on that edge.

It is convenient to start a problem with the initial moments and reactions as those of the panels having zero slopes and zero deflections at all edges. These forces will be called "fixed-edge moments" and "fixed-edge reactions", respectively. These should be determined by using the same network as that used in the determination of the distribution constants.

In general, these fixed-edge moments and fixed-edge reactions are not statically balanced at all the joints. The problem is to find a compatible set of deformations at all joints in the structure such that the moments and reactions at all of these joints are in equilibrium.

(a.) Sign Conventions

The sign conventions used in the distribution procedure are as follows:

Clockwise moments acting on a plate and clockwise rotations are positive. Downward vertical forces acting on a plate and downward deflections are positive.

Deformations and forces on an E-W edge are viewed from the east and those on an N-S edge are viewed from the south.

Another way of looking at these sign conventions is to consider forces acting on a plate that produce positive deformations as positive; where positive rotations are clockwise and positive deflections are downward.

(b.) Effects of Relaxing a Moment at a Joint

When an unbalanced moment, M_u, (Fig. 6b), which is the algebraic sum of the moments on the two bars adjoint with i, is released, the edge A is assumed to have temporary supports against deflections but free to rotate. The relaxation of M_u at i will produce a rotation, φ_i , sufficient to provide equilibrium of moments at i. This will also induce reactions at the other joints on edge A for both panels L and R, and also moments and reactions on edges B, C, D and B', C', D'. All these forces are found as follows:

For equilibrium of moments at joint i, (Fig. 6b);

$$\delta M_i^L + \delta M_i^R + M_u = 0 \tag{15}$$

For continuity, the rotations should be:

$$\varphi_i^L = \varphi_i^R = \varphi_i \; .$$

Since

$$\varphi_i^L = \frac{\delta M_i^L}{K_i^L} \quad \text{and} \quad \varphi_i^R = \frac{\delta M_i^R}{K_i^R}$$

$$\frac{\delta M_i^L}{K_i^L} = \frac{\delta M_i^R}{K_i^R} \tag{16}$$

Equations (15) and (16) are sufficient to determine uniquely the values of δM_i^L and δM_i^R and therefore also φ_i.

The other forces can be found from their relationships with φ_i given in Section 13a with $\varphi_{ii} = \varphi_i$, which are:

$$m_{ji} = k_{ji} \cdot \varphi_i \; ; \; (m_{ji} \text{ on edge A are zero})$$

and

$$r_{ji} = q_{ji} \cdot \varphi_i$$

and the rotations on edge A are:

$$\varphi_{ji} = \Omega_{ji} \cdot \varphi_i$$

Continuity and equilibrium of moments at all other joints on edge A are also preserved when i is balanced in moment.

By the principle of superposition, the total forces and rotations at j after relaxing and balancing the moments at all the joints on edge A are:

$$\overline{m}_j = \sum_{i=1}^{m} k_{ji} \cdot \varphi_i \; ; \; (\overline{m}_j \text{ on edge A are zero})$$

and

$$\overline{r}_j = \sum_{i=1}^{m} q_{ji} \cdot \varphi_i$$

and

$$\overline{\varphi}_j = \sum_{i=1}^{m} \Omega_{ji} \cdot \varphi_i$$

The values of φ_{ji} have to be recorded for each joint relaxed, since the final boundary rotations, θ_j, will be the summation of all such φ_{ji}, or

$$\theta_j = \sum_{c=1}^{d} \overline{\varphi}_j^c$$

$$= \sum_{c=1}^{d} \sum_{i=1}^{m} \Omega_{ji} \cdot \varphi_i^c$$

where: d = number of cycles required for convergence to a desired degree of accuracy.

(c.) Effects of Relaxing a Reaction at a Joint

When an unbalanced reaction, R_u, (Fig. 6c), which is the algebraic sum of the reactions at i of the two panels adjoint with i, is released, the edge A is assumed to have temporary support preventing rotations, but free to deflect. The relaxation of R_u at i will produce a deflection δ_i sufficient to provide equilibrium of vertical forces at i. This will also induce moments at the other joints on edge A for both L and R panels, and also moments and reactions on edges B, C, D and B', C', D', which are found as follows:

For equilibrium of vertical forces at joint i, (Fig. 6c),

$$\delta R_i^L + \delta R_i^R + R_u = 0 \tag{17}$$

and for continuity, the deflections should be:

$$\delta_i^L = \delta_i^R = \delta_i$$

Since $\delta_i^L = \dfrac{\delta R_i^L}{T_i^L}$ and $\delta_i^R = \dfrac{\delta R_i^R}{T_i^R}$

$$\frac{\delta R_i^L}{T_i^L} = \frac{\delta R_i^R}{T_i^R} \tag{18}$$

Equations (17) and (18) will determine δR_i^L and δR_i^R and therefore also δ_i .

The other forces are therefore:

$$m_{ji} = q'_{ji} \cdot \delta_i$$

and

$$r_{ji} = t_{ji} \cdot \delta_i \; ; \; (r_{ji} \text{ on edge A are zero})$$

and the deflections on edge A are:

$$\delta_{ji} = \psi_{ji} \cdot \delta_i \; .$$

Continuity and equilibrium of vertical forces at all other joints on edge A are also preserved when i is balanced in reaction.

The total forces and deflections at j after relaxing and balancing the reactions at all the joints on edge A are:

$$\bar{m}_j = \sum_{i=1}^{m} q'_{ji} \cdot \delta_i$$

$$\bar{r}_j = \sum_{i=1}^{m} t_{ji} \cdot \delta_i \; ; \; (\bar{r}_j \text{ on edge A are zero})$$

and

$$\bar{\delta}_j = \sum_{i=1}^{m} \psi_{ji} \cdot \delta_i$$

The values of δ_{ji}, likewise, have to be recorded for each joint relaxed, since the final boundary deflections, Δ_j, are:

$$\Delta_j = \sum_{c=1}^{d} \bar{\delta}_j^c$$

$$= \sum_{c=1}^{d} \sum_{i=1}^{m} \psi_{ji} \cdot \delta_i^c$$

It should be noted that when a joint is balanced in vertical forces, the moments do not in general balance; likewise, the vertical forces do not balance when a joint is balanced in moments. However, for panels with the same q_{ji} and q'_{ji}, which is the case for continuous plates of identical panels, these forces are also balanced.

15. Programming for the ILLIAC*

The solution for a continuous plate, in terms of deflections, consists of two steps:

(i) determination of the edge deformations;

(ii) determination of w_c from the edge deformations of (i) and adding $w_c + w_p$.

Both these steps are to be done on the ILLIAC*. The write-ups of the complete programs for step (i) are given in Appendix B-1, and for step (ii) in Appendix B-2. The corresponding flow charts may be found in Appendix A.

The moments and shears (including reactions) at all points in every panel can be computed by a program coded for this purpose, which is not presented here.

The deflection coefficients, α_i and β_i, and the distribution factors, K_i, T_i, k_{ji}, etc., have to be previously determined before undertaking steps (i) and (ii). These factors and coefficients have to correspond to each aspect ratio of the panels and the beam stiffnesses of the problem. The network used should be such that the edges of the columns coincide with the network. These are found by finite difference equations and solved by programs for the automatic solution of simultaneous equations, such as Program L7-230 of the ILLIAC Library.

* The ILLIAC is an automatic electronic digital computer with 1024 highspeed memory and 12,800 drum memory. The approximate time required for an arithmetical operation are: addition, 90 μsec.; multiplication, 700 μsec; division, 800 μsec. For more information about the ILLIAC, the reader is referred to Ref. 17.

The present programs are for problems involving identical panels in a structure. However, extension of these programs to more general cases can be made without any difficulty.

IV. ACCURACY OF SOLUTIONS

16. General Remarks

Since this method of analysis is, in essence, an extension of the piecewise solution of single-panel plates by the finite difference method, the inaccuracies inherent in the approximation by finite differences are also found in this method. Maximum accuracy of solutions by this method would be that of the finite difference solutions, which would be obtained when total convergence has been achieved. However, the method is such that, without much difficulty, a very fine network which will assure a good degree of accuracy can be used.

The programs coded for the ILLIAC are such that the degree of convergence for a problem can be specified beforehand.

17. Comparisons with Available Solutions

There are, unfortunately, no available exact solutions for comparison with the results given by this method for the types of problems presented. The only direct comparison is made for solutions by this method and by a direct finite difference method for a typical interior panel of a "flat plate".

(a.) Typical Interior Panel--The solution for a typical interior panel of a "flat plate", which is a panel within a structure having an infinite number of identical panels loaded uniformly throughout, can be obtained by direct application of finite difference equations. The solution for a typical square interior panel by direct finite differences is presented by the author in Ref. 18. The column dimensions were taken as 1/10 of the panel length and a 20 by 20 network was used, which made the edges of the columns coincide with the network. The same problem is solved by this method in Section 18.

The deflections, with convergence carried to eight places, are compared with the corresponding deflections by the finite difference method in Table I.

(b.) Accuracy of Distribution Factors--There is no exact way of determining the distribution factors; neither is there any available information for direct comparison with the values obtained here. These values, however, can be compared approximately with solutions of plates fixed on all edges by use of the reciprocal theorem.

Referring to Figs. 7a and 7b, the following relationship can be derived by the reciprocal theorem:

$$M_i \cdot \varphi_i = \underline{z_{1i}}\, p \tag{19}$$

where $\underline{z_{1i}}$ is the deflection matrix due to φ_i. This relationship is approximate because of the presence of the columns in Fig. 7a. Taking i at the mid-point of the edge

$$M_i = -0.0513\ pL^2 \qquad \text{(from Ref. 8, Table 27 with } b/a = 1\text{)}$$

therefore, $$M_i\varphi_i = +0.0513\ pL^2$$

since $\varphi_i = 1$.

For this case,

$$\underline{z_{1i}} \cdot p = +0.0506\ pL^2$$

The approximate relationship, Eq. (19), is therefore satisfied. The difference is of the right order because the presence of the columns gives smaller deflections, $z_{1i}(x,y)$, than otherwise would be obtained.

V. ILLUSTRATIVE PROBLEMS

The solutions to a few problems of continuous plates are given in the following sections. These are intended only to show the applicability of the method in handling problems of these types.

The plates shown in the following problems are supported on columns only, and are thus similar to the type of reinforced concrete floor commonly referred to as "flat plate", with $b/a = 1$. Problems of this type, without beams, will in general give the slowest convergence. The addition of beams will hasten the convergence and a structure with all beams infinitely stiff in either torsion of flexure will give the fastest convergence.

A 20 by 20 network has been used in all of the following problems.

18. Problem I.

A typical square interior panel of a "flat plate" structure is analyzed by this method in the following manner:

A square plate with all edges fixed and loaded uniformly is analyzed by finite differences to obtain the fixed-edge moments and fixed-edge reactions of the plate. This involves the determination of the deflections of only 1/8 of a panel of the plate because of symmetry. With these fixed-edge moments and fixed-edge reactions as the initial forces, the reactions only are relaxed and distributed. This is done by Code CP-19, Appendix B-1. When the residual reactions are zero, or less than a specified tolerable error, the corresponding deflections are the edge deflections of the typical interior panel with the edge rotations remaining zero. The deflections of the interior points are obtained by the relationships given in Section 10 with the use of Code SM-19, Appendix B-2.

These deflections are presented in Table I, with the convergence carried out to eight decimal places. This degree of accuracy for the edge deflections requires about ten cycles of calculations for this particular problem, which takes about 20 minutes of the ILLIAC time.

19. Problem II.

A more realistic structure used in practice is presented. The structure, Fig. 8, is composed of nine panels supported on columns only. Panels 2, 5, and 8 are loaded uniformly with an intensity p. All other panels are not loaded.

The deflections and moments across certain sections of the plate are presented in Figs. 10 through 13.

20. Problem III.

The same structure in Problem II is analyzed for panels 1, 4, and 7 loaded and no load on all other panels, Fig. 9. The corresponding deflections and moments across certain sections of the plate are presented in Figs. 14 through 17.

By superposition, the results of Problems II and III can be used to obtain solutions for other combinations of loading.

Each of Problems II and III takes about two hours of calculation time on the ILLIAC for an accuracy of eight decimal places on the edge rotations and edge deflections.

VI. SUMMARY

A method for the analysis of rectangular plate continuous in two directions over torsionally and flexurally stiff beams and rectangular columns or columns only is presented. The method is essentially an extension of the use of finite differences to the analysis of continuous plates by the interconnection of piecewise single-panel solutions. This extension is made possible by representing a plate with Newmark's plate analog, which reduces the mathematical problem to a problem of finding a solution to a physical model, hence facilitates the fulfillments of physical continuity and equilibrium between adjoining panels when interconnecting the different panels within a continuous structure.

The method of analysis is based on the principle of superposition in which the complete solution of a panel within a continuous plate can be expressed as,

$$w = w_p + w_c$$

where; w_p = a solution satisfying the equation $\nabla^2 \nabla^2 w = p/N$, and an arbitrarily prescribed boundary condition for the panel,

w_c = a solution satisfying the equation $\nabla^2 \nabla^2 w = 0$, and the boundary deformations of the panel which together with the boundary conditions of w_p satisfy the conditions of equilibrium and continuity with the other panels of the structure.

The partial solution w_p is determined by direct application of finite difference equations to a panel. It is convenient to prescribe the boundary conditions with zero slopes and zero deflections at all the edges of the panel. This will involve 1/4 of the panel for rectangular panels and 1/8 of the panel for square panels.

The boundary deformations are determined by a distribution procedure, the essential features of which are based on Cross's moment distribution procedure for beams and frames. This procedure relaxes, in succession, one force at a joint on an edge at a time and allowing the other joints on the same edge to deform in a prescribed fashion such that continuity and equilibrium of at least one force are maintained over the entire edge. The deformations of the edge, either rotations or deflections, are added algebraically to the corresponding previous values whenever a force at a joint is relaxed. The final edge rotations, θ_j, and edge deflections, Δ_j, of every panel corresponding to a set of residual moments and reactions whose absolute values are all less than a tolerable error, are the boundary deformations of all the panels in a continuous structure.

The partial solution w_c is determined from the boundary deformations by the following relationships:

$$w_c(x,y) = \sum_{j=1}^{2(m+n)} [\alpha_j(x,y)\ \theta_j + \beta_j(x,y)\ \Delta_j]$$

where j is taken for all the joints on the perimeter of a panel. The coefficients $\alpha_j(x,y)$ and $\beta_j(x,y)$ are the deflections at an interior point (x,y) due, respectively, to a unit edge rotation and unit edge deflection at j.

All calculations that are involved in the determination of w are done on the ILLIAC (the University of Illinois digital computer). Programs for this purpose have been coded in two main parts: one of these is to determine the boundary deformations by the distribution procedure with the zero edge rotations and zero edge deflections and the corresponding fixed-edge moments and fixed-edge reactions of all panels given as original data; the other part computes w_c by use of the above equation and forms $w_p + w_c$ of all interior points for one panel at a time with the boundary deformations and w_p for the corresponding panel given as data. A third program has been coded (which is not presented), which computes the moments and shears (including the reactions) at all points from the given w.

The maximum accuracy of the method of analysis is the same as that of the finite difference method. This is obtained when total convergence has been achieved; a comparison between a solution by this method and a solution by direct finite differences for a typical square interior panel of a flat plate verifies this conclusion, as shown in Table I. However, the method is such that, without much difficulty, a very fine network which will assure a good degree of accuracy can be used.

Several problems are presented to show the applicability of the method in handling problems of these types. A typical square interior panel and two nine-panel flat-plate structures with different loading conditions, Figs. 8 and 9, has been analyzed. The results of the latter problems, in terms of the deflections and moments across certain sections of the structure, are presented in Figs. 10 through 17.

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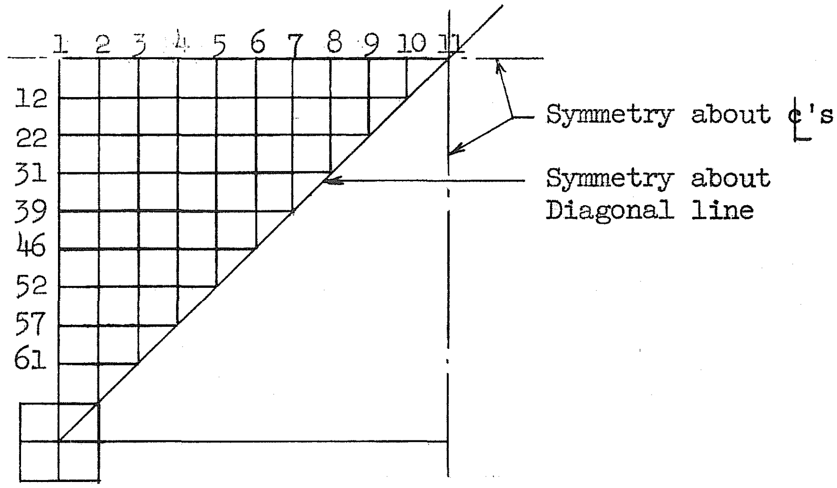


TABLE I

Points	Deflections, in terms of $\frac{WL^2}{200\ N}$	
	By Direct Finite Differences	By Distribution Procedure
1	0.60832741	0.60832741
2	0.61526100	0.61426100
3	0.63531006	0.63531006
4	0.66632168	0.66632168
5	0.70503018	0.70503018
6	0.74746996	0.74746996
7	0.78943087	0.78943087
8	0.82688578	0.82688578
9	0.85634849	0.85634849
10	0.87514531	0.87514531
11	0.88160092	0.88160092
12	0.59546618	0.59546618
22	0.55750484	0.55750484
31	0.49634842	0.49634842
39	0.41531340	0.41531340
46	0.31940499	0.31940499
52	0.21588671	0.21588671
57	0.11548110	0.11548110
61	0.03484218	0.03484218

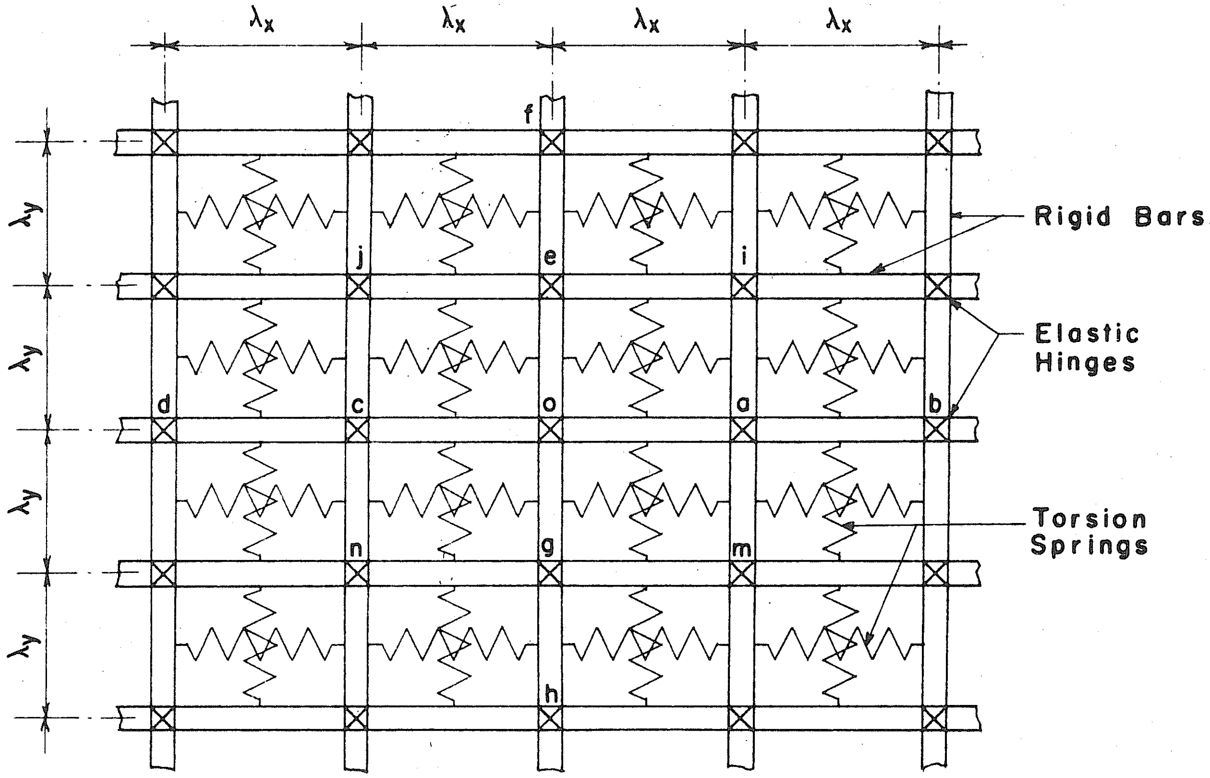


FIG. 1a: NEWMARK'S PLATE ANALOG

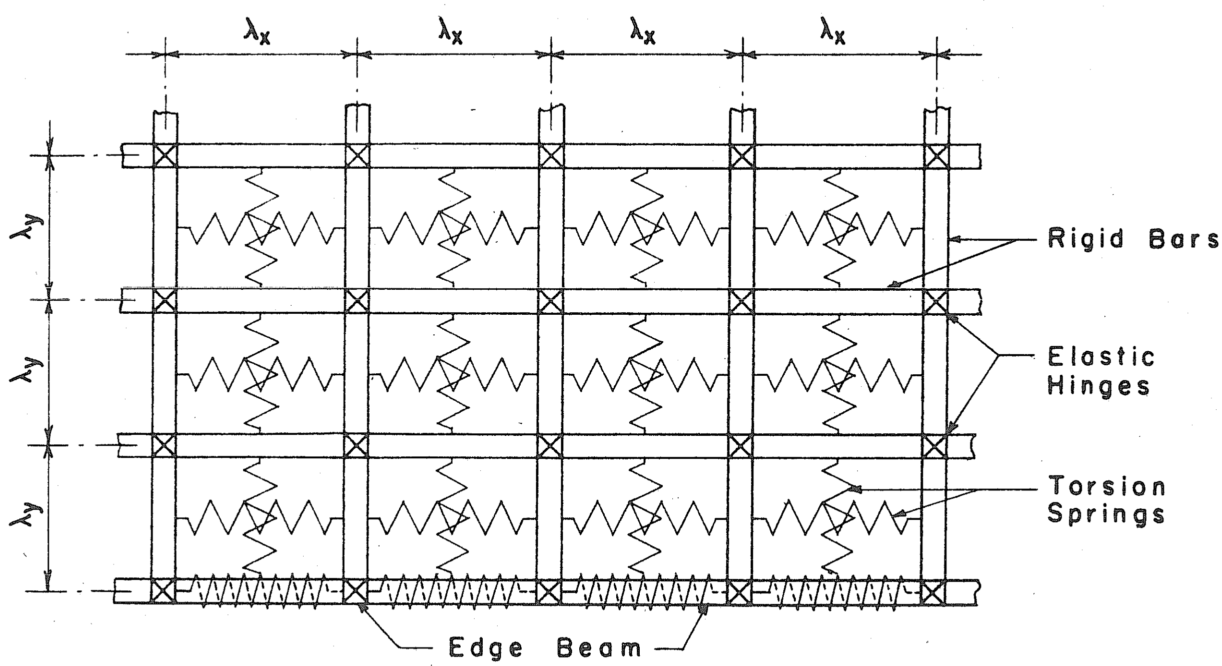


FIG. 1b: NEWMARK'S PLATE ANALOG WITH EDGE BEAM

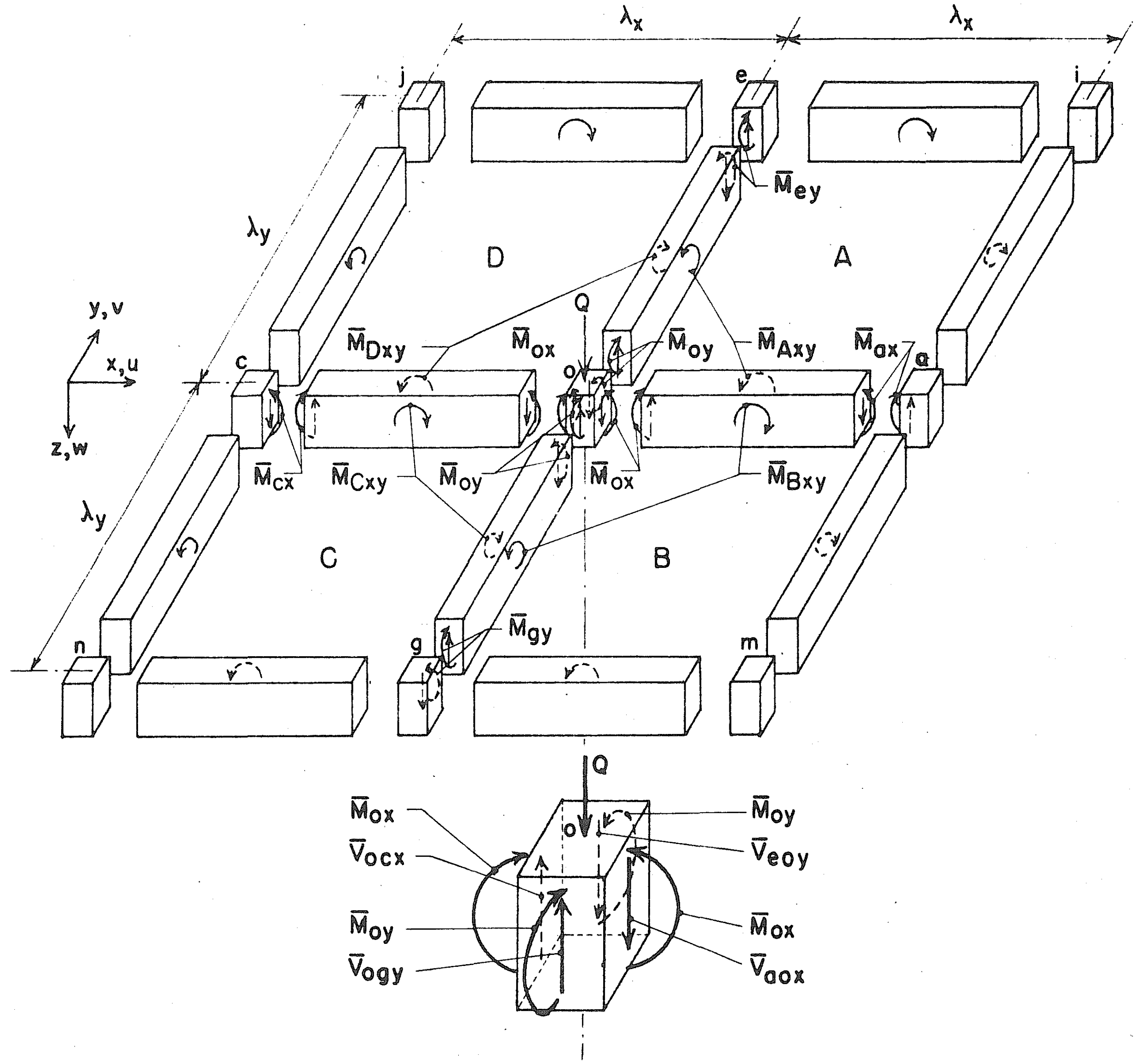


FIG. 2a: FORCES AFFECTING EQUILIBRIUM OF HINGE O

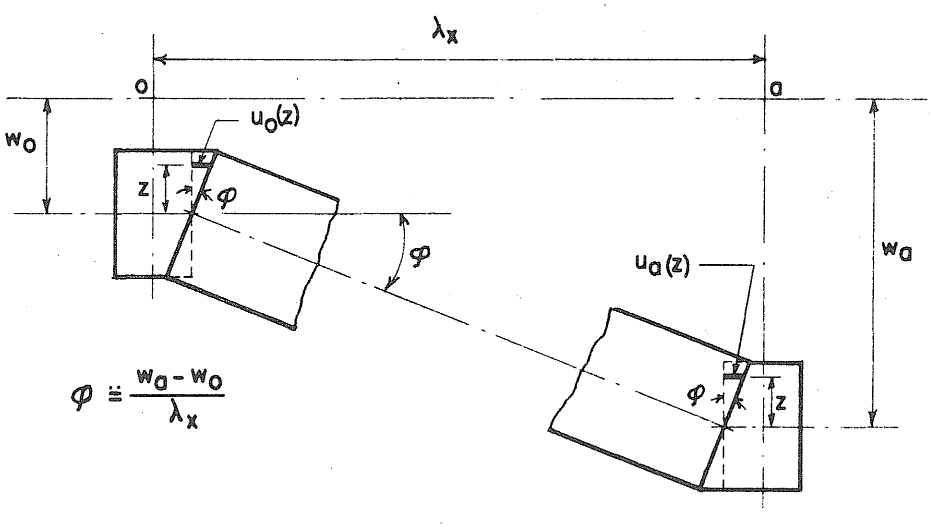


FIG. 2b: RELATION BETWEEN u and w

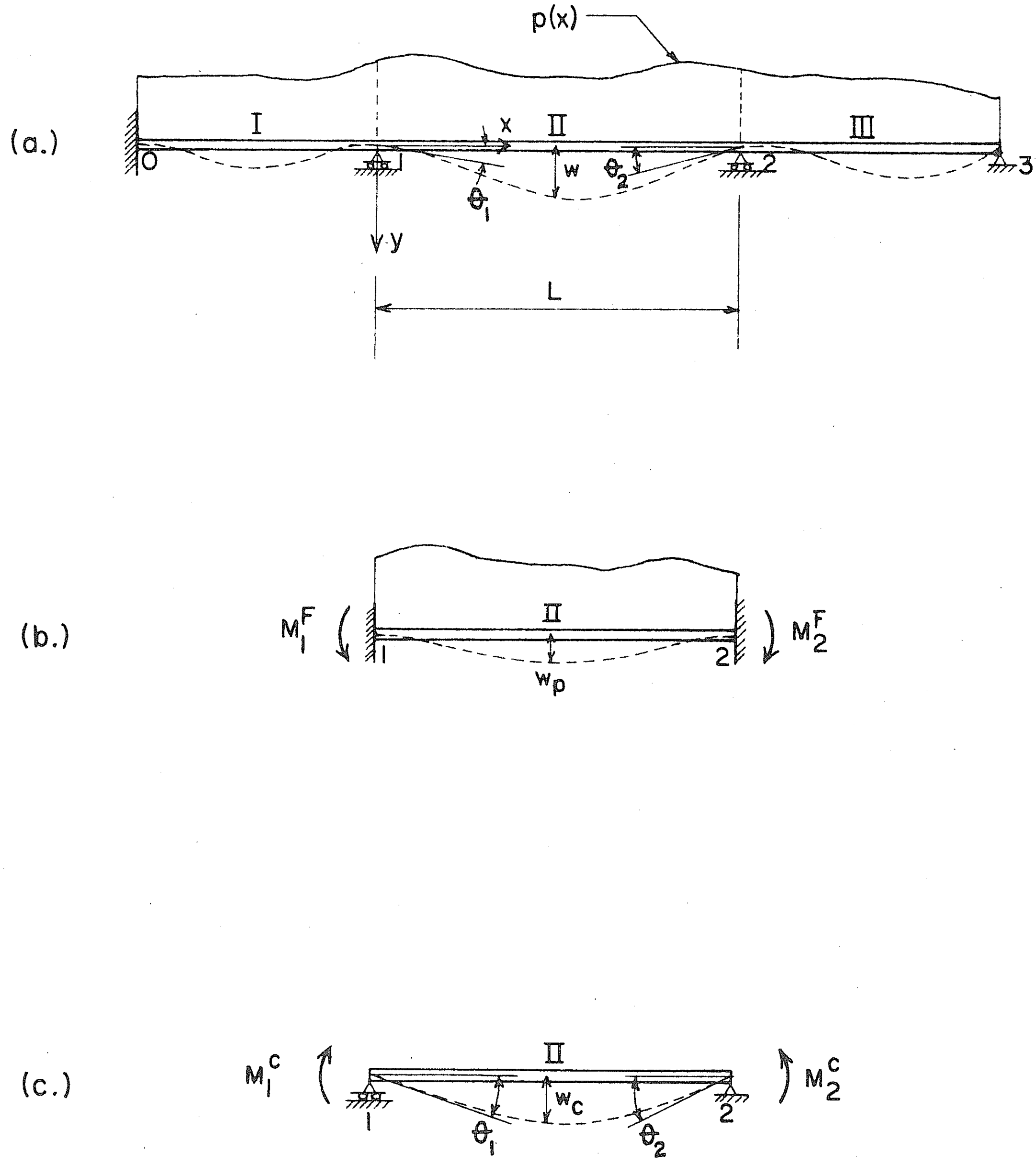
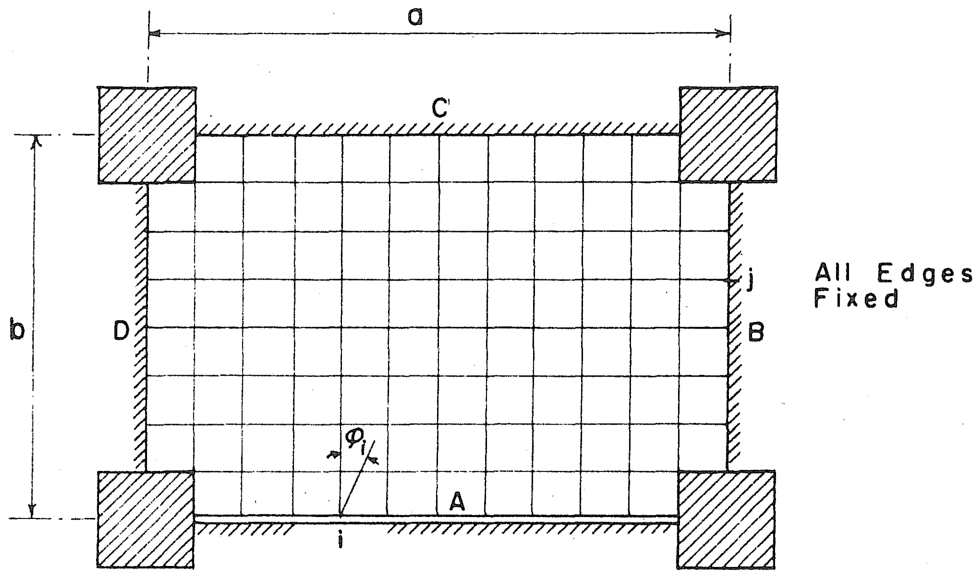


FIG. 3: A CONTINUOUS BEAM

(a.)



(b.)

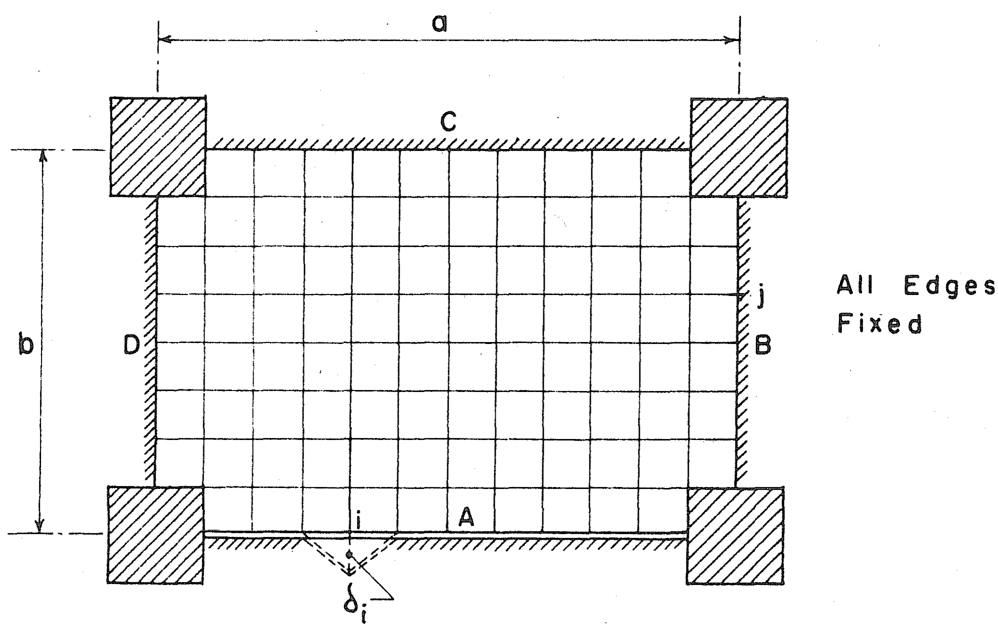
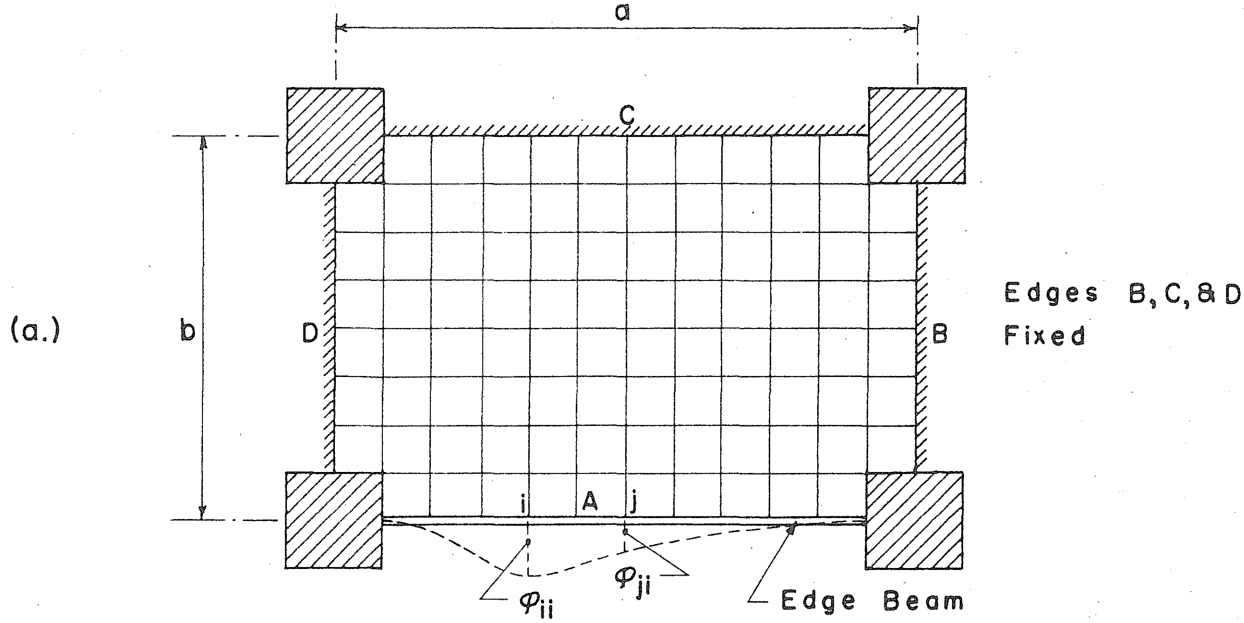


FIG. 4: DEFORMATIONS AT A SINGLE JOINT i



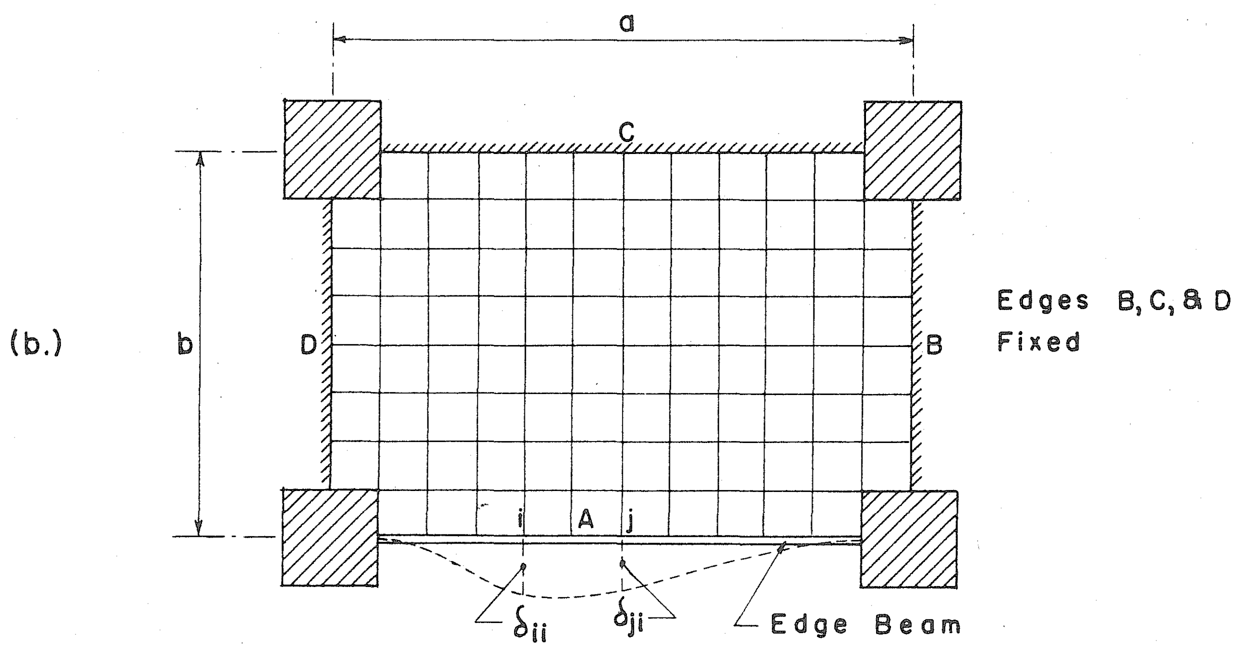


FIG. 5: DEFORMATIONS ON AN EDGE.

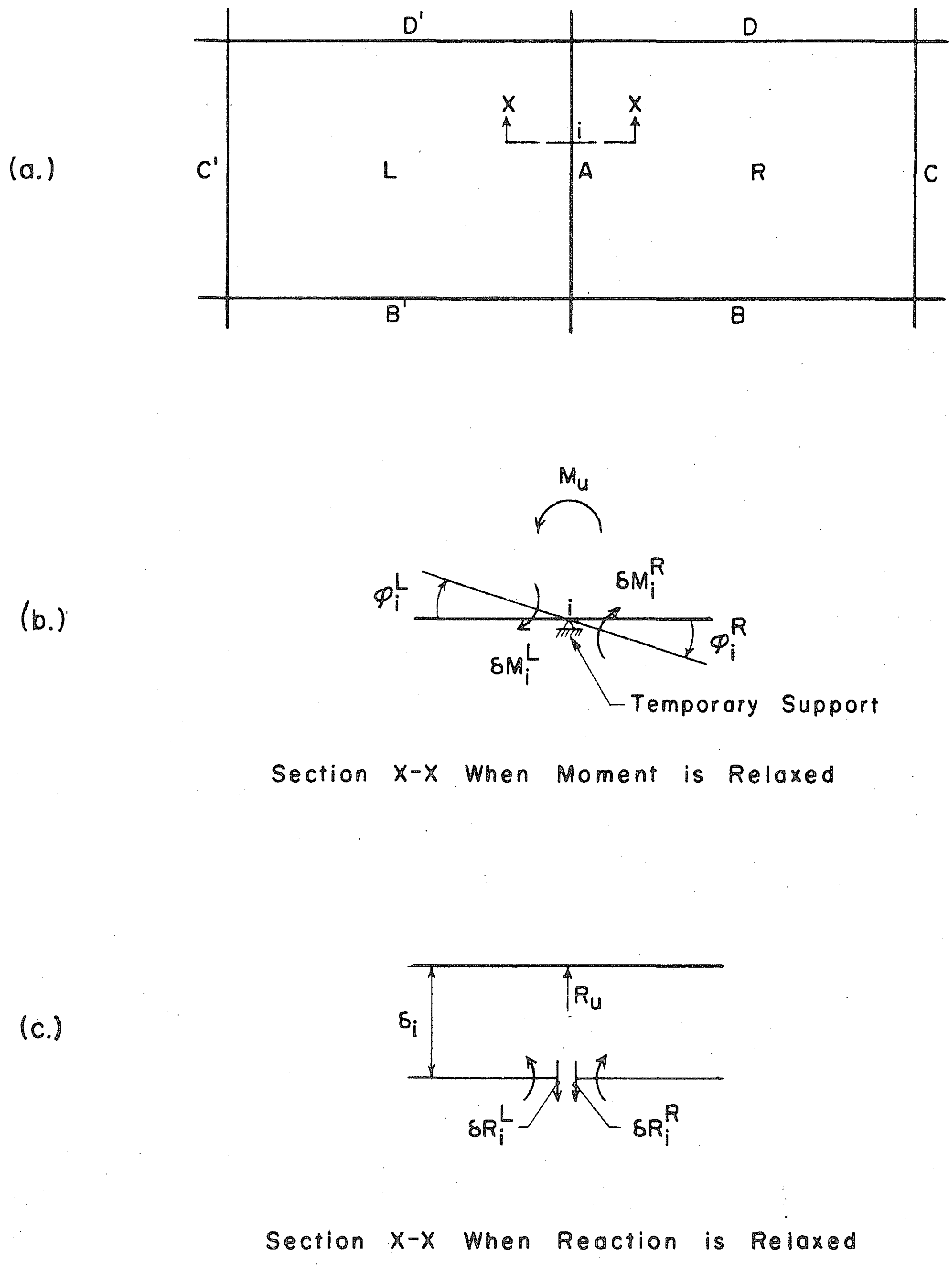


FIG. 6: RELAXATION OF FORCES AT A JOINT

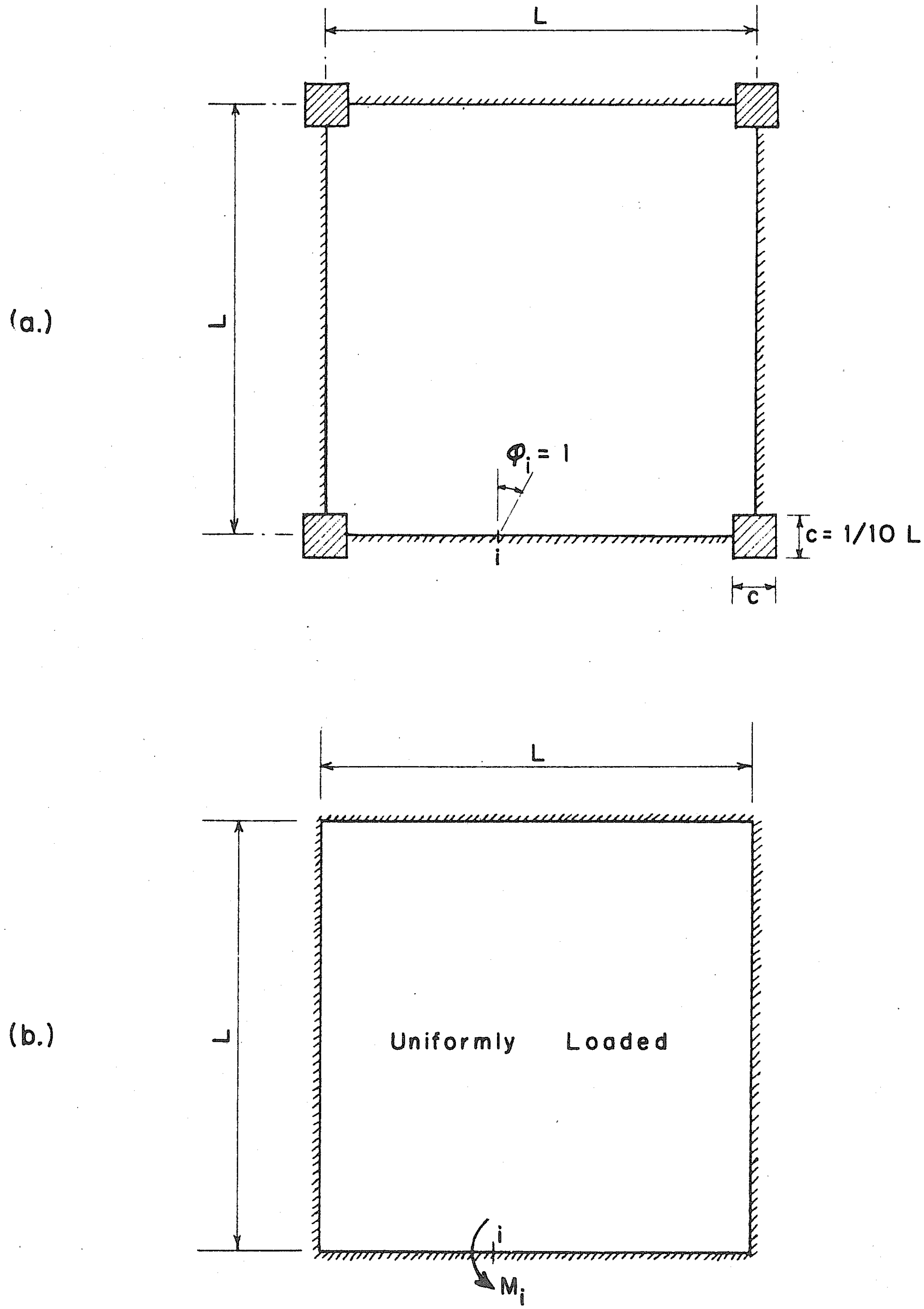
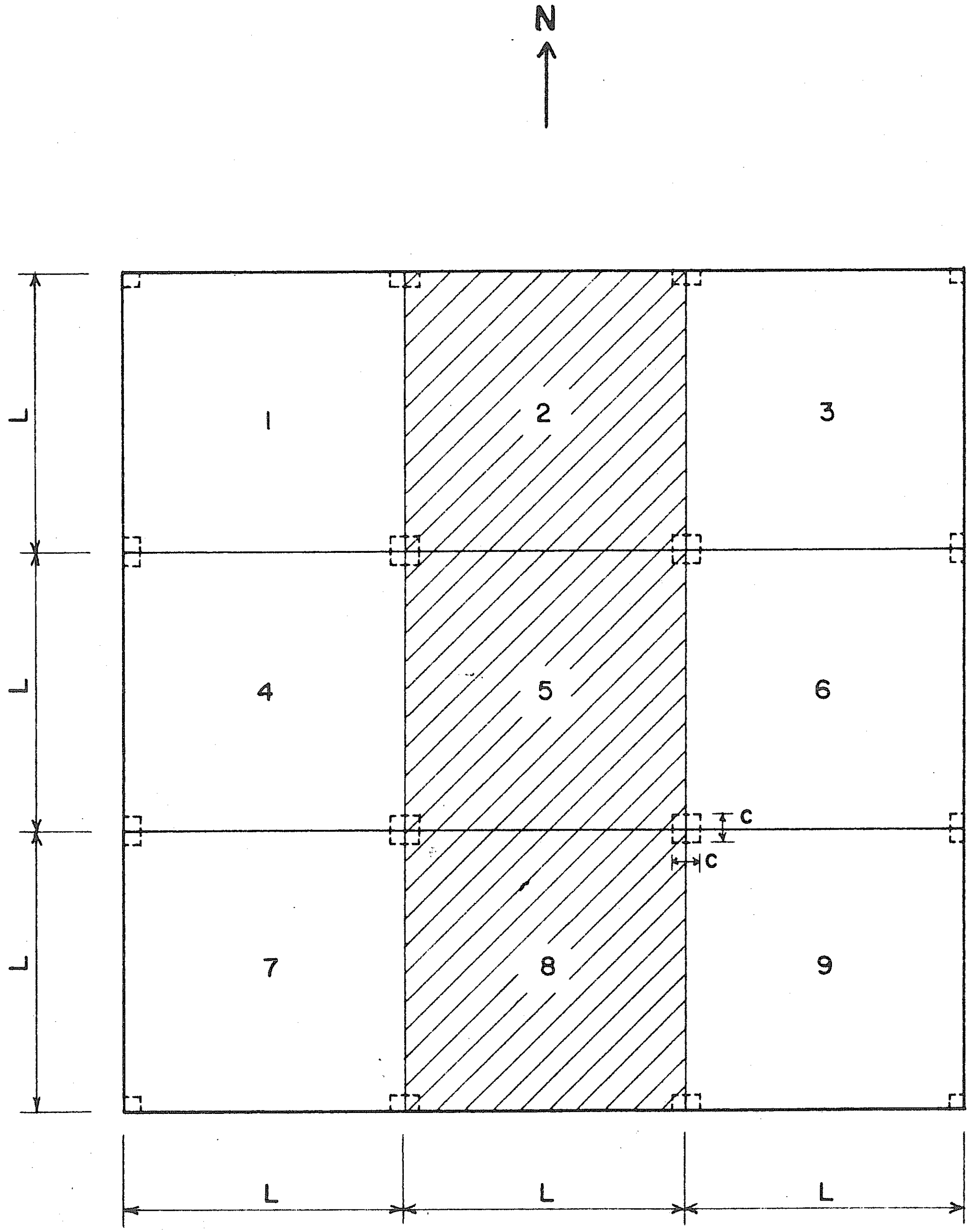


FIG. 7: ACCURACY OF DISTRIBUTION FACTORS BY RECIPROCAL THEOREM

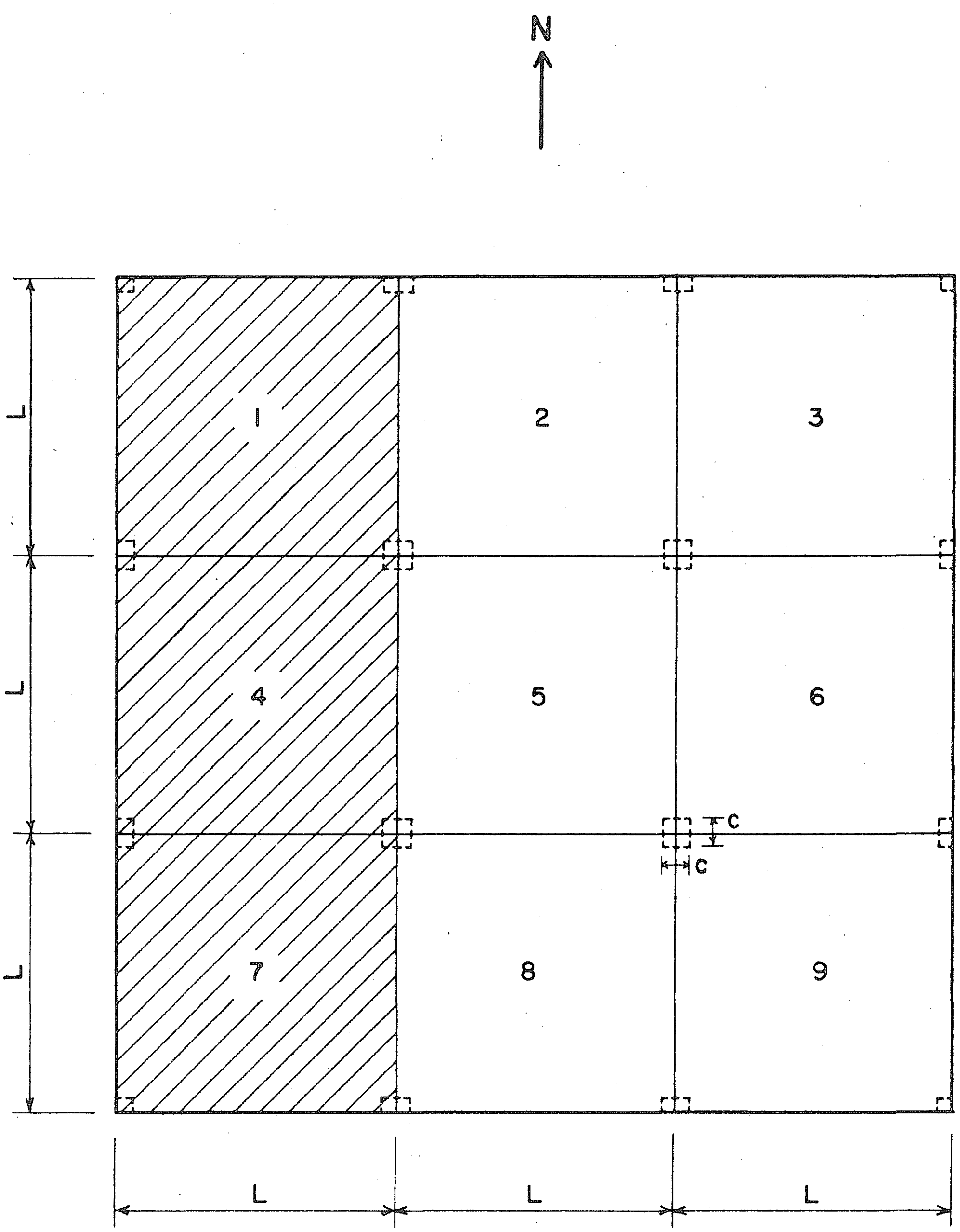


Panels 2, 5, & 8 Uniformly Loaded

Total Load per Loaded Panel = W

$c/L = 1/10$

FIG. 8: NINE-PANEL STRUCTURE -- CENTER STRIP LOADED



Panels 1, 4, & 7 Uniformly Loaded

Total Load per Loaded Panel = W

$c/L = 1/10$

FIG. 9: NINE-PANEL STRUCTURE -- SIDE STRIP LOADED

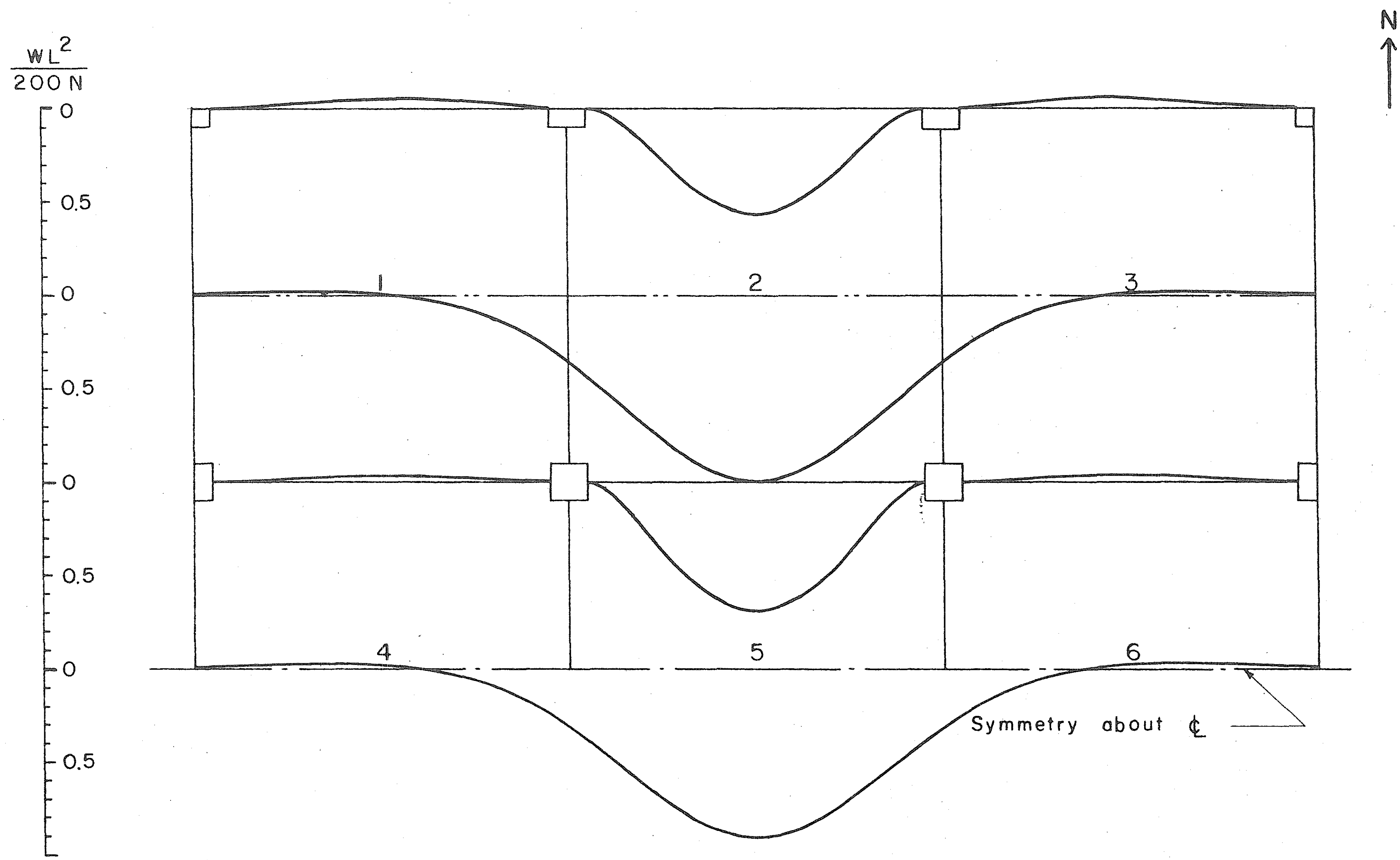


FIG. 10: DEFLECTIONS AT SECTIONS SHOWN — PANELS 2, 5, & 8 LOADED

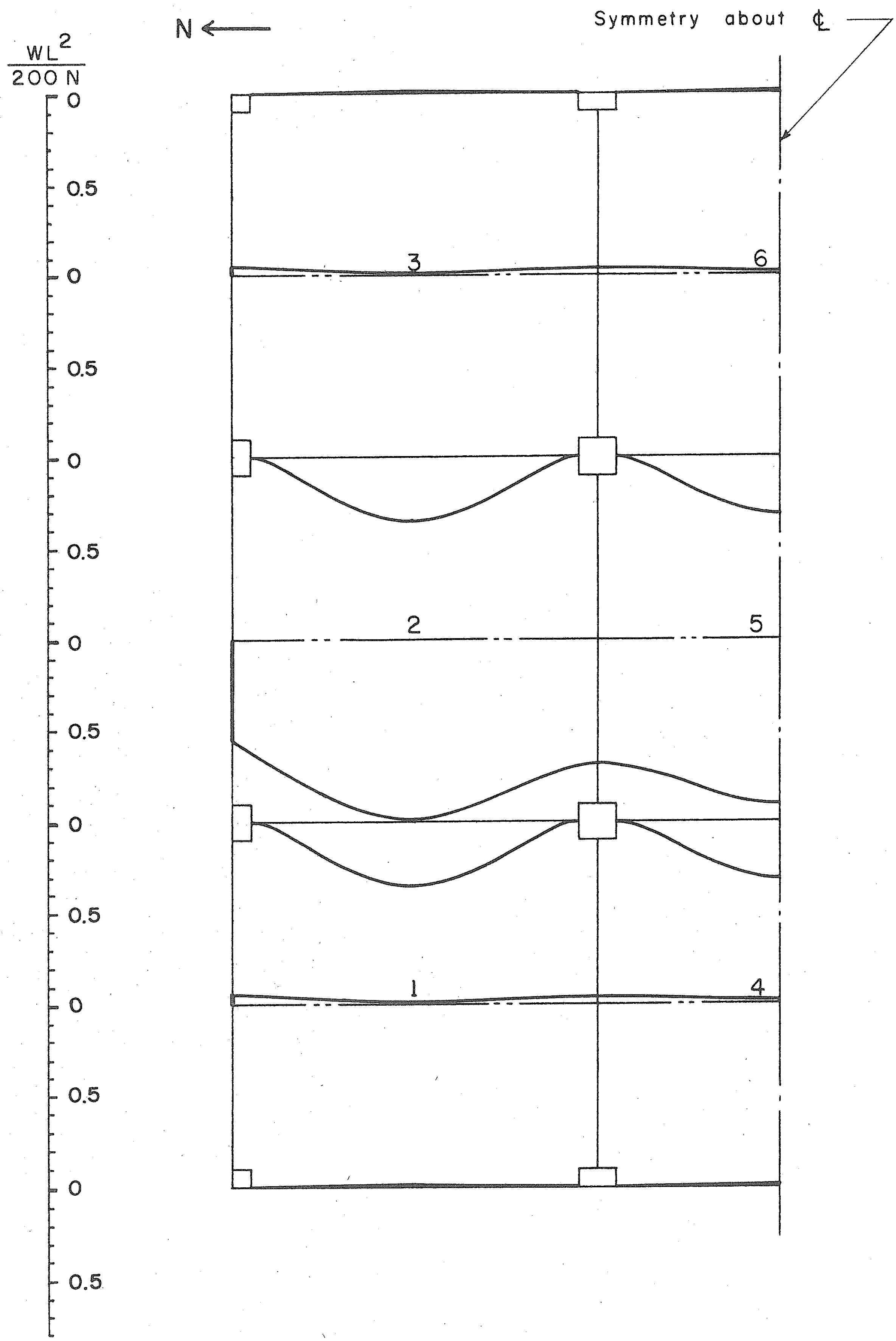


FIG. 11: DEFLECTIONS AT SECTIONS SHOWN
PANELS 2, 5, & 8 LOADED

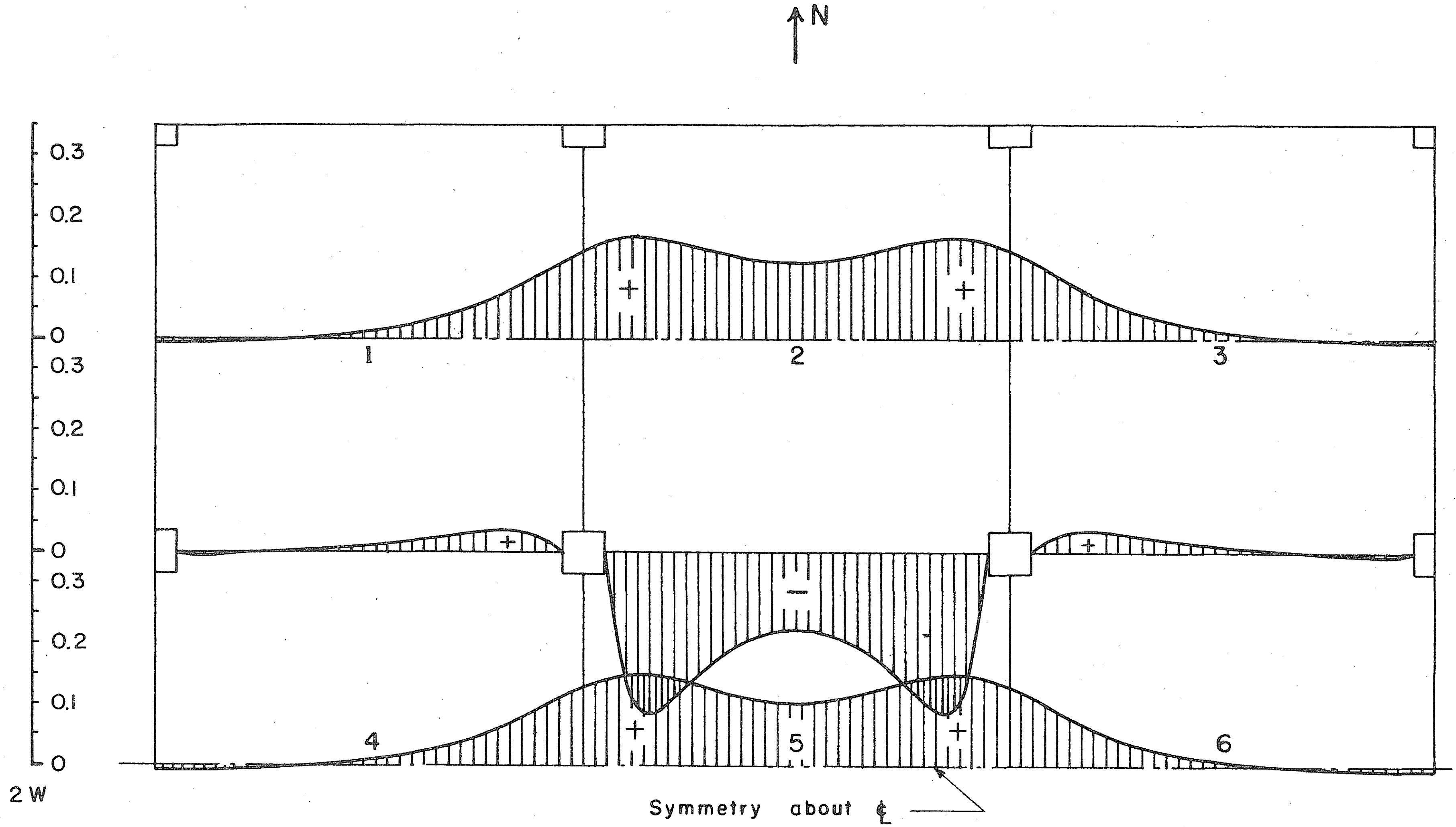


FIG. 12: BENDING MOMENTS ACROSS SECTIONS SHOWN — PANELS 2, 5, & 8 LOADED

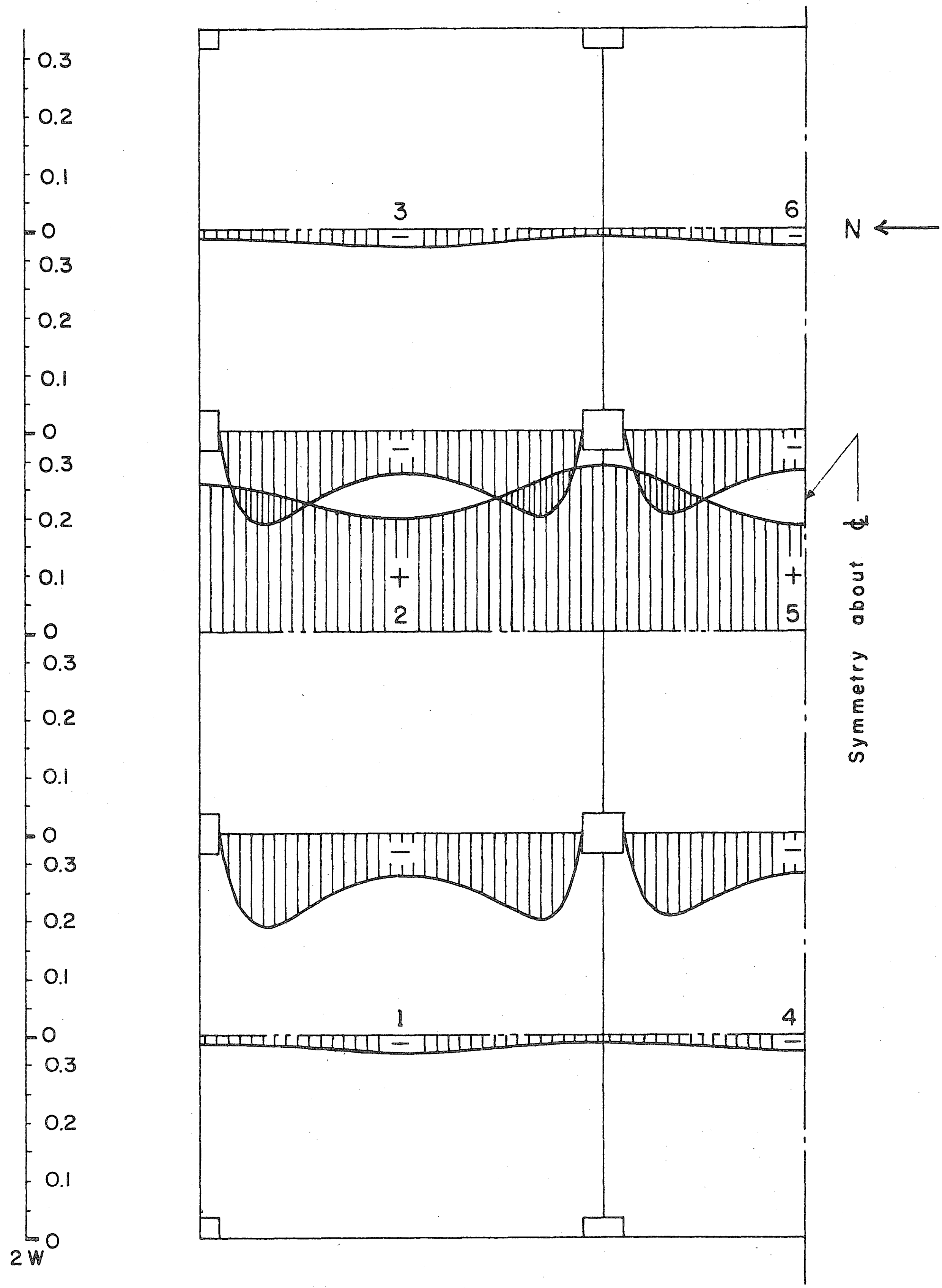


FIG. 13: BENDING MOMENTS ACROSS SECTIONS SHOWN
PANELS 2, 5, & 8 LOADED

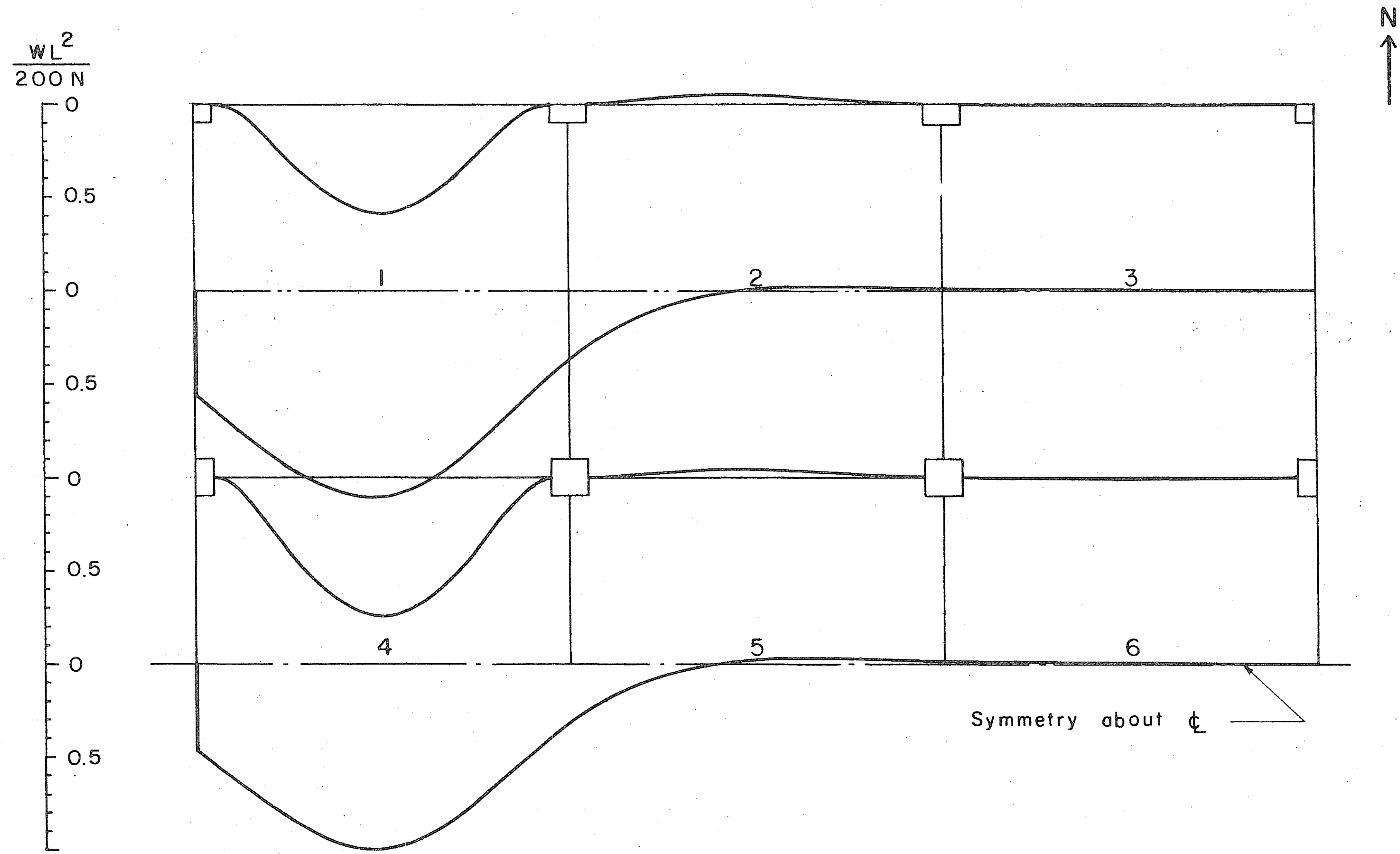


FIG. 14: DEFLECTIONS AT SECTIONS SHOWN — PANELS 1, 4, & 7 LOADED

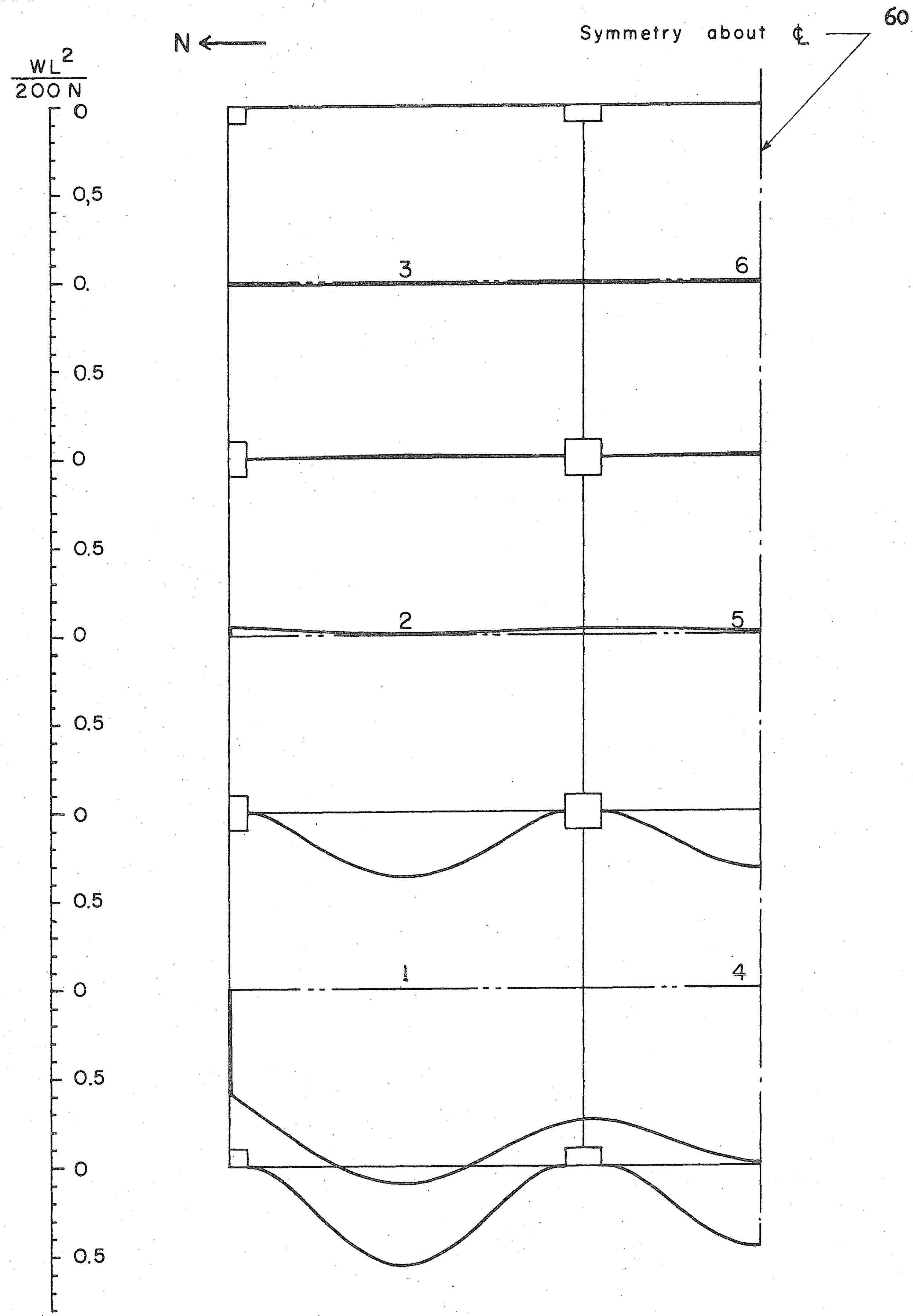


FIG. 15: DEFLECTIONS AT SECTIONS SHOWN

PANELS 1, 4, & 7 LOADED

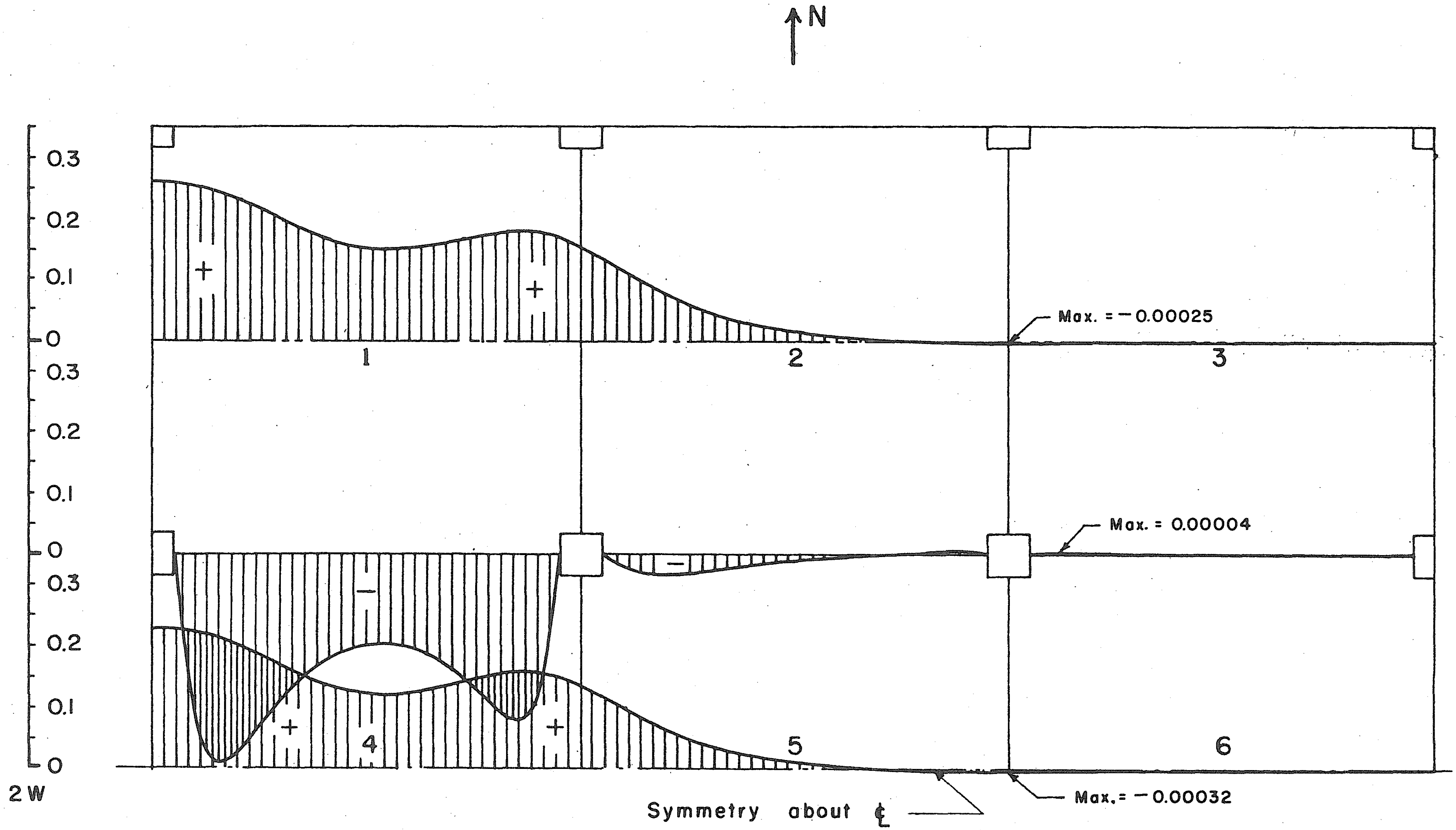


FIG. 16: BENDING MOMENTS ACROSS SECTIONS SHOWN — PANELS 1, 4, & 7 LOADED

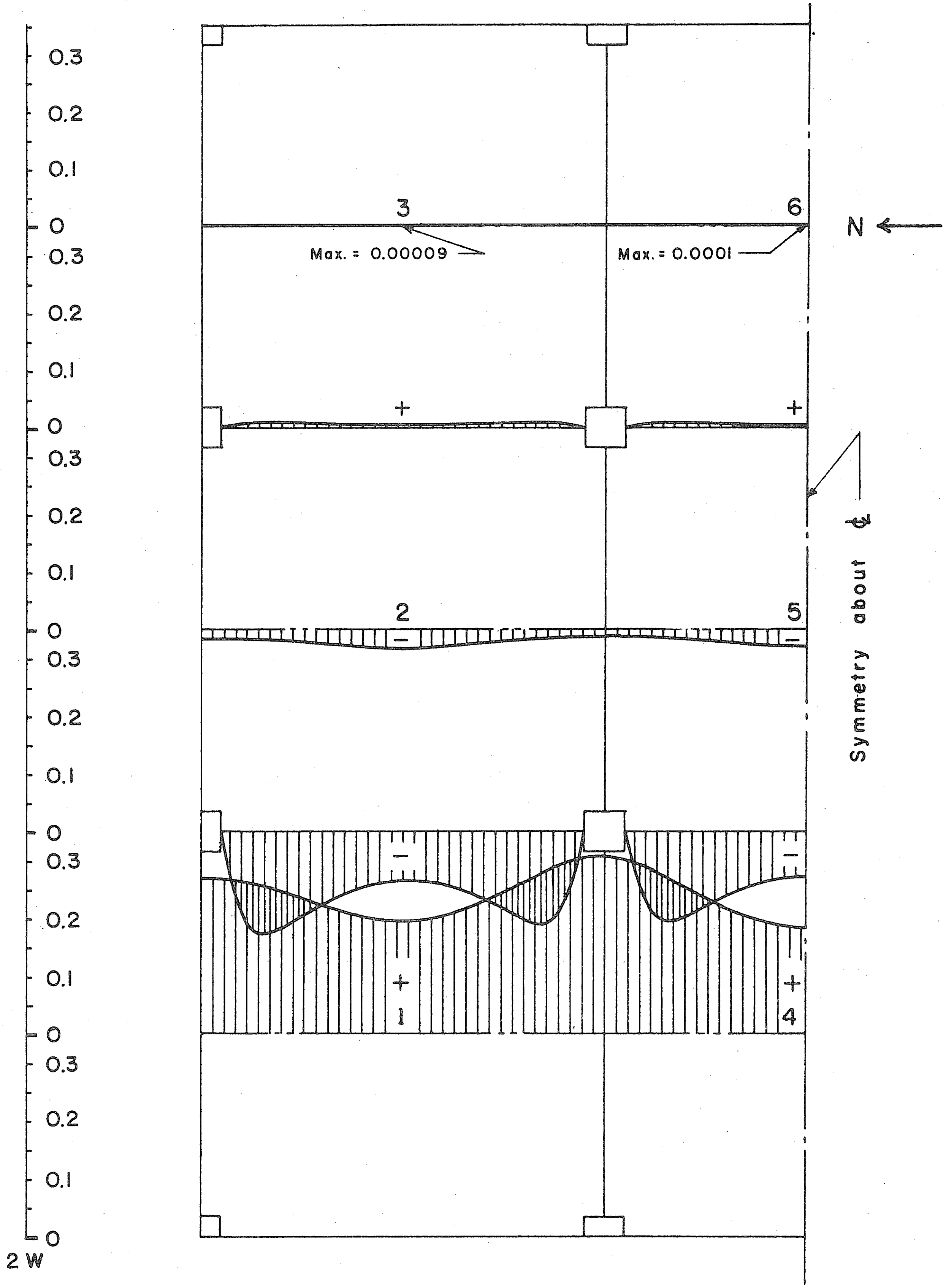


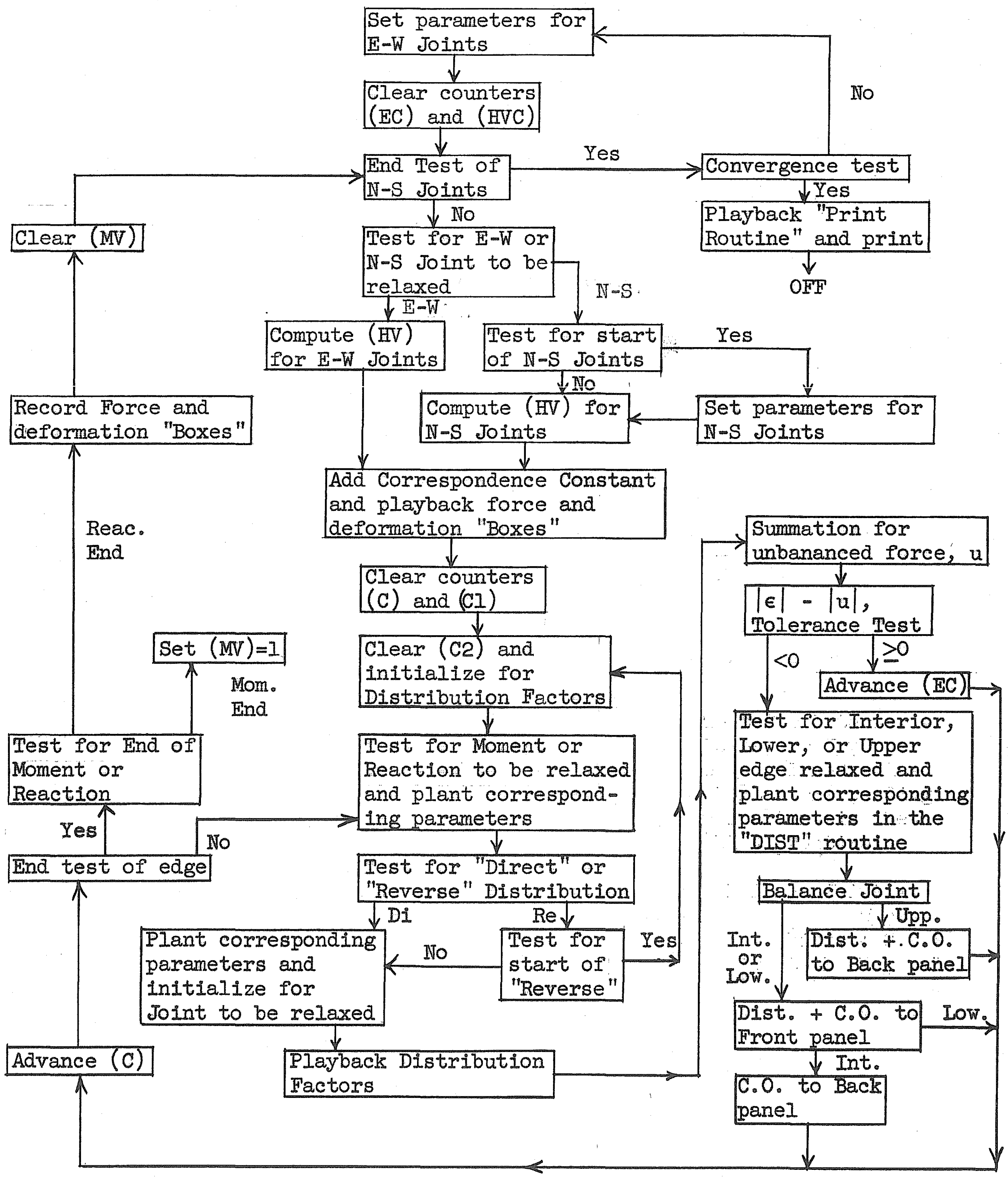
FIG. 17: BENDING MOMENTS ACROSS SECTIONS SHOWN
PANELS 1, 4, & 7 LOADED

APPENDIX A

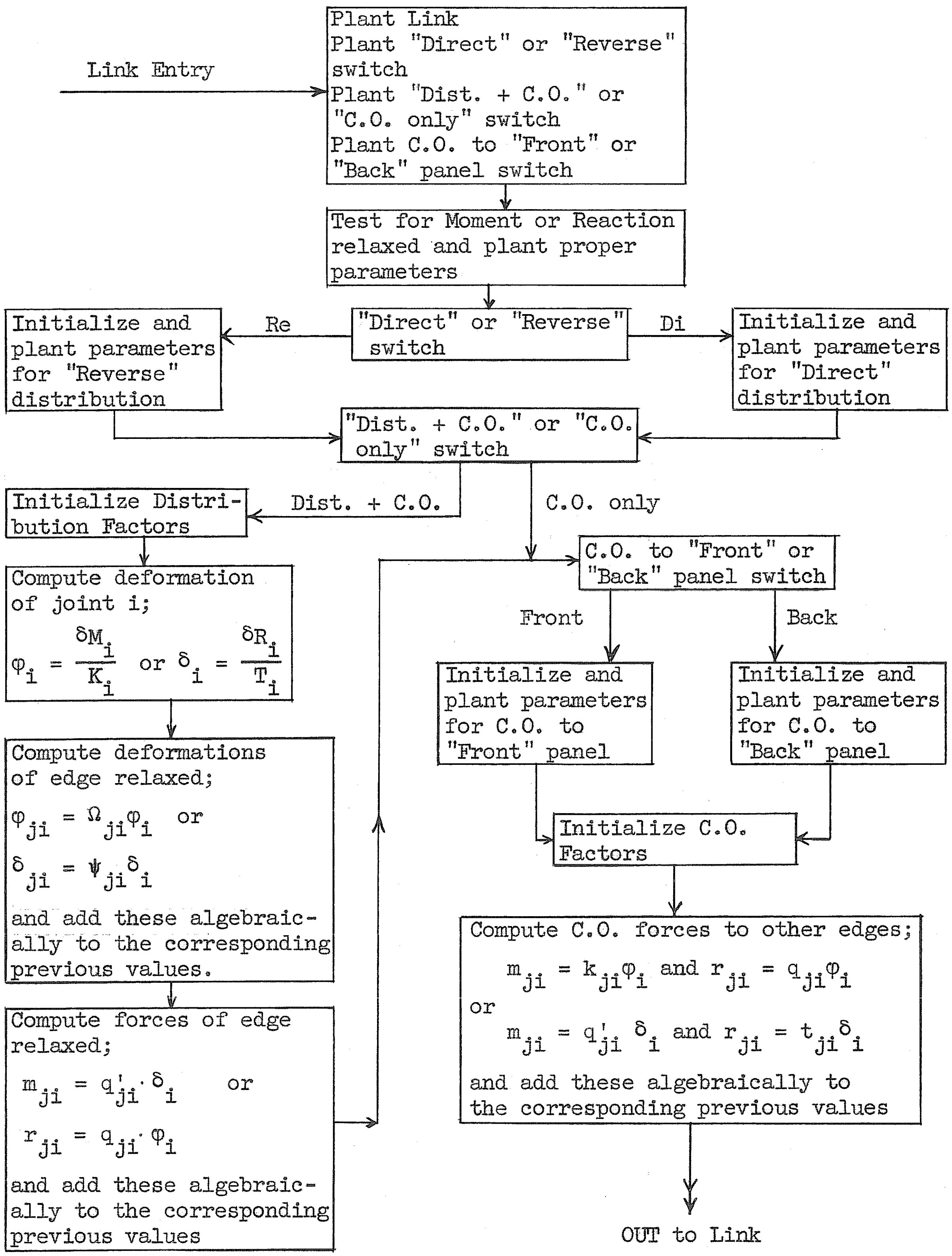
FLOW DIAGRAMS

A-1. Code CP-17:

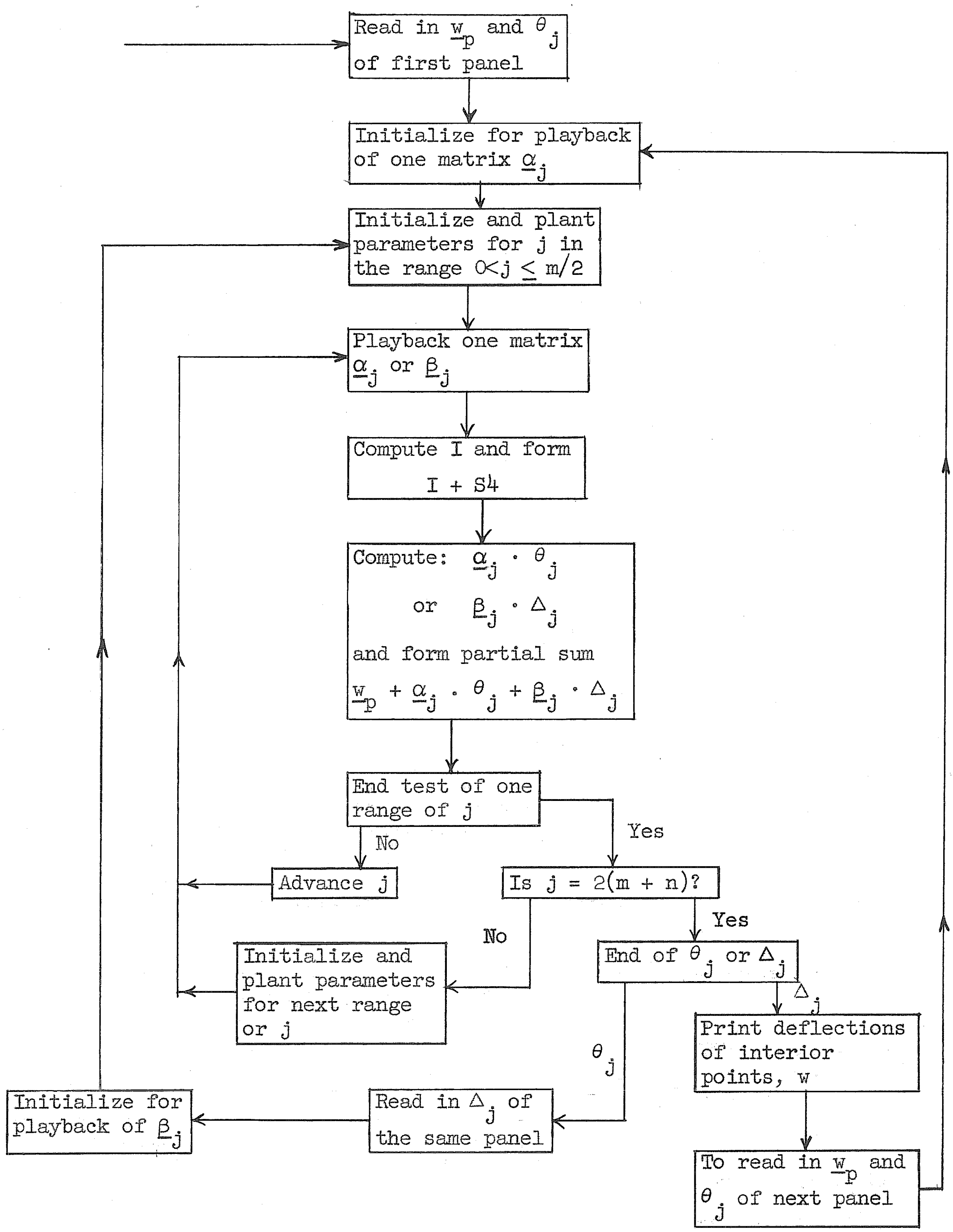
(a.) Control



(b.) Distribution Routine



A-2. Code SM-19



Read in $\underline{w}_p$ and θ_j of first panel
Initialize for playback of one matrix $\underline{\alpha}_j$
Initialize and plant parameters for j in the range $0<j \leq m/2$
Playback one matrix $\underline{\alpha}_j$ or $\underline{\beta}_j$
Compute I and form I + S4
Compute: $\underline{\alpha}_j \cdot \theta_j$ or $\underline{\beta}_j \cdot \Delta_j$ and form partial sum $\underline{w}_p + \underline{\alpha}_j \cdot \theta_j + \underline{\beta}_j \cdot \Delta_j$
End test of one range of j
No
Yes
Advance j
Is $j = 2(m + n)$?
No
Yes
Initialize and plant parameters for next range or j
End of θ_j or Δ_j
Δ_j
θ_j
Print deflections of interior points, w
Initialize for playback of $\underline{\beta}_j$
Read in Δ_j of the same panel
To read in $\underline{w}_p$ and θ_j of next panel

APPENDIX B

PROGRAM WRITE-UPS

B-1. Code CP-17

Title --- Determination of Edge Deformations by Moment and Reaction Distribution

Type --- Complete program

Number of Words --- 826

Duration --- $\frac{5}{9}(N_I + 0.6N_E)(\frac{m + n}{34})$ minutes per cycle.

where: N_I = number of interior edge.

N_E = number of exterior edge.

Pre-set Parameters:

S3 --- Tolerable error

S4 --- m, the number of joints in an E-W edge

S5 --- n, the number of joints in an N-S edge

S6 --- h, the number of panels in the east-west direction

S7 --- v, the number of panels in the north-south direction

S8 --- number of Distribution Factors per joint.

These pre-set parameters define completely the geometry of a problem.

Sub-routines:

N12 --- Infraput
P16 --- Infraprint
Y1 --- Drum transfer

} ILLIAC Library

DIST --- Distribution Routine (See Flow Diagrams and Orders)

HV --- To compute reference joints

PR and PR1 --- To playback and record force and deformation "Boxes"

PT and PT1 --- Print Routine

IN --- Input Routine (an interlude); to input distribution factors and initial data.

William's Memory Assignments:

0, 1, 2 --- Temporary storage

3 - 8 --- Pre-set parameters

9 - 68 --- Constants, counters, and parameters

69 - 108 --- Y1

109 - 136 --- HV

137 - 172 --- PR1

173 - 237 --- PR

238 - 406 --- DIST

407 - 535 --- "CONTROL"

540 - 676 --- Distribution factors for a joint

(540 - 670, used by Input Routine during input)

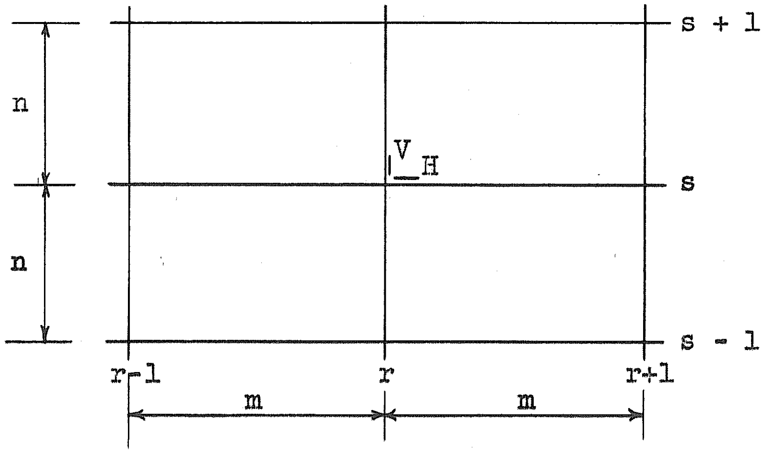
(540 - 701, used by Print Routine during output)

680 -1019 --- Temporary storage for force and deformation "Boxes" during computations.

Description:

This program computes the unbalanced force at a joint; distributes this unbalanced force to the panels connected with the joint; computes the deformation of the joint due to this distributed force and computes the forces and deformations of the edge. The carry-over forces (moments and reactions) to the other edges of the panels under calculations are computed and all these forces and deformations are added algebraically to their corresponding initial values. This is done for one joint at a time and the whole process is repeated until the unbalanced forces at every joint in a given structure are less than a tolerable error specified in S3.

The following reference system is used:



where:

m = number of joints in an E-W edge

n = number of joints in an N-S edge

h = number of panels in the east-west direction

v = number of panels in the north-south direction

s = the number of the E-W edge

r = the number of the N-S edge

The E-W edges are numbered from bottom to top and the N-S edges from left to right. A panel is referred to as the (s,r)-panel if its south-west corner is the intersection of the s-edge and the r-edge.

The joints on the E-W edges are numbered consecutively from left to right, followed by the joints on the N-S edges, which are numbered from bottom to top. A joint in the (s,r)-panel is determined from its position with respect to a reference joint (H,V), where

H = the first joint of the lower E-W edge of the (s,r)-panel

V = the first joint of the left N-S edge of the (s,r)-panel,

which are:

$$H = (2s - 1)hm + (r - 1)m + 1$$

$$V = (24 - 1)vn + (s-1)n + 1 + 2(v + 1)hm$$

Form of Output:

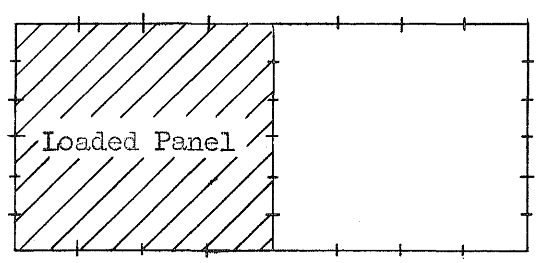
The edge deformations are printed to the number of digits corresponding to the tolerable error specified in S3. The edge rotations of a panel are printed in a single column in consecutive order followed by the edge deflections. Each deformation starts with the joint H and goes counterclockwise to joint V and ending with the character N. The edge rotations are preceded by a heading "ROTATIONS" and the edge deflections by "DEFLECTIONS".

Preparation of Data:

The initial values of the rotations and deflections of every joint in a structure, which are zeros, and its corresponding fixed-edge moments and fixed-edge reactions should be given. Zeros are typed with a "+". Clockwise moments and downward reactions are positive. These initial forces and deformations are prepared in blocks of m or n, terminating each block with the character N in the following order: rotations, moments, deflections, and reactions.

The values on the E-W edges are typed first, starting with those on the lowest edge, in a left-to-right order. These are followed by the values on the N-S edges, starting with the left-most edge, in a bottom-to-top order.

For example:



with: m = 3; n = 5;

and using: M = fixed moment at j; and R = fixed reaction at j; the initial data should appear as follows:

Rotations: +++N +++N +++N +++N +++N +++N +++N +++N

+++++N +++++N +++++N +++++N +++++N +++++N

Moments: +++N +++N MMMN +++N MMMN +++N +++N +++N

+++++N MMMMMN MMMMMN +++++N +++++N +++++N

Deflections: +++N +++N +++N +++N +++N +++N +++N +++N

+++++N +++++N +++++N +++++N +++++N +++++N

Reactions: +++N +++N RRRN +++N RRRN +++N +++N +++N

+++++N RRRRRN RRRRRN +++++N +++++N +++++N

(a.) <u>Routine --- CONTROL</u>

Loc.	Orders	Notes
	(CONT)	
0	41 (MV)	Clear counters
	41 (EC)	
1	41 (HVC)	
	F5 (HVC)	Test for End of N-S Joints
2	40 (VC)	
	FO (222)	
3	32 5L	
	L5 (HVC)	Test for E-W or N-S Joints relaxed
4	FO (111)	
	36 32L	
5	22 17L	
	L5 (END)	Test for Convergence
6	FO (EC)	
	34 8L	
7	26 126L	→ To playback "Print" routine
	50 F	waste
8	L5 (1)	← START
	40 (S)	s = 1
9	41 (R)	r = 0
	L5 4F	Set parameters for E-W Joints
10	40 (PM)	
	00 20F	
11	46 (M-1)	
	L5 5F	
12	00 20F	
	46 (N-1)	

```
13    50  (m)
      75  6F
14    S5  F
      40  (HM)
15    50  (N)
      75  7F                                                For E-W Joints relaxed
16    S5  F
      40  (VN)
17    22  0L    → To Clear counters
      F5  (R)
18    40  (R)           Test for Right side
      F0  6F
19    36  20L
      26  39L   ——→     To compute HV for
20    F5  (S)           E-W Joints
      40  (S)
21    L5  (1)
      40  (R)
22    22  19L
      L5  (1)
23    40  (R)         r = 1
      41  (S)         s = 0
24    L5  5F
      40  (PM)
25    00  20F
      26  (M-1)
26    L5  4F
      00  20F
27    46  (N-1)
      50  (N)           Set parameters for
28    75  7F            N-S Joints
      S5  F
29    40  (HM)
      50  (M)
30    75  6F
      S5  F                                                 For N-S Joints relaxed
31    40  (VN)
      22  33L
```

32	F5	(111)	
	L0	(HVC)	
33	32	22L	
	F5	(S)	
34	40	(S)	Test for Upper side
	F0	7F	
35	36	36L	
	26	42L	→ To compute HV for N-S Joints
36	F5	(R)	
	40	(R)	
37	L5	(1)	
	40	(S)	
38	22	35L	
	50	F	waste
39	50	(REF)	
	50	39L	Compute HV for E-W Joints
40	26	(HV)	
	50	(AUX)	
41	26	44L	
	50	F	waste
42	50	(AUX)	
	50	42L	Compute HV for N-S Joints
43	26	(HV)	
	50	(REF)	
44	50	(PLAY)	Playback Force and Deformation "Boxes"
	50	44L	
45	26	(PR1)	
	41	(C)	
46	L3	(1)	Set and clear counters
	40	(C1)	
47	41	(C2)	
	L3	(MV)	Test for Mom. or Reac. relaxed
48	36	50L	
	L5	(T)	
49	40	71L	
	26	51L	Initialize drum address for Distribution Factors
50	L5	(+)	
	40	71L	

```
51    L3  (MV)        Test for Mom. or Reac.
      36  54L         relaxed
52    L5  (1FF)
      40  14F         Reaction relaxed
53    L5  (3FF)
      22  55L
54    L5  (3FF)
      40  14F         Moment relaxed
55    L5  (1FF)
      40  13F
56    46  62L
      46  67L
57    F5  (PM)
      10  1F          Test for "Direct" or
58    F0  (C)         "Reverse"
      32  65L
59    F5  (C1)
      40  (C1)        Test for Start of
60    L3  (C1)        "Reverse"
      36  47L    -->  To clear (C2) ....
61    L5  141(DIST)
      42  15F
62    L5  F          by 56
      L4  (M-1)
63    L4  (M-1)                          Initialize for "Reverse".
      L0  (01)
64    L0  (C2)
      40  (F101)
65    22  68L
      L5  97(DIST)
66    42  15F
      L5  (M-1)
67    L4  F          by 56'           Initialize for
      L4  (C2)                        "Direct"
68    40  (F101)
      L5  (C2)
```

```
69      L4  (01)
        40  (C2)
70      50  540F
        50  70L
71      26  (Y1)
        00  F             Playback Distribution
72      00  S8            Factors for Joint i
        L5  71L
73      F4  8F
        40  71L
74      L5  (F101)
        46  79L
75      50  F
        46  103L
76      46  104L
        L4  (2-1)
77      46  80L
        50  F
78      46  105L
        46  106L
79      L1  F       by 74'
        50  F       waste      Summation for unbalanced
80      L0  F       by 77      Force, u
        40  0F
81      L7  3F
        L2  0F      |ε| - |u|
82      36  124L    ---> To advance (EC)
        L5  (REF)
83      L0  (MAX)         Test for E-W or
        32  87L           N-S Joints relaxed
84      L1  (S)
        L4  (1)
85      32  97L   ---> To Lower edge
        L5  (S)
86      F0  7F
        32  99L   ---> To Upper edge
```

```
87    22  90L   --> To Interior edge         Test for Interior,
      L1  (R)                                Lower, or Upper edge
88    L4  (1)
      32  97L   --> To Lower edge
89    L5  (R)
      F0  6F
90    32  99L   --> To Upper edge
      L5  0F
91    10  1F        -1/2 u at 0 and 1
      40  0F
92    40  1F
      L5  112L
93    42  111L                               Plant parameters for
      L5  109L                               Interior edges
94    46  112L
      L5  105L
95    42  106L
      26  103L
96    L3  (MV)
      32  122L         Test for Ed of Mom. or Reac.
97    26  120L
      41  1F    --  Clear N(1)
98    L5  119L      N(0) = -u        Plant parameters for
      42  111L                       Lower edges
99    22  94L
      L5  0F        Clear N(0)
100   40  1F        N(1) = -u
      41  0F
101   L5  112L                               Plant parameters for
      42  106L                               Upper edges
102   L5  107L
      46  112L
103   L5  F         by 75'
      L4  0F
104   40  F         by 76
      L5  1F                         Balance Joint
105   L4  F         by 78
      50  107L      waste
```

```
106   40  F          by 78'
      26  F          by 95,101'
107   51  21(DIST)
      50  107L
108   26  (DIST)
      00  13F                              "Dist + C.O." to
109   51  66(DIST)                         "Front" panels
      50  109L
110   26  (DIST)
      33  14F
111   50  F          waste
      26  F          by 93,98',
112   5S  F          by 94,102'
      50  112L
113   26  (DIST)
      00  13F
114   L1  1F                               "Dist + C.O." or
      40  1F                               "C.O. only" to "Back"
115   5S  66(DIST)                          panels
      50  115L
116   26  (DIST)
      33  14F
117   F5  (C)
      40  (C)           Advance (C) and End test of edge
118   L0  (PM)
      36  96L
119   26  51L      Loop
      50  117L
120   50  (RE)
      50  120L     Record Force and
121   26  (PR1)    Deformation "Boxes"
      41  (MV)
122   22  1L
      L5  (1)
123   40  (MV)
      22  45L   →  To clear (C), (C1), etc.
```

```
124     F5   (EC)
        40   (EC)
125     26   117L   →  To advance (C)
        50   F         waste
126     50   540F
        50   126L
127     26   (Y1)         Playback "Print" routine
        00   2600F
128     00   148F
        26   (PT)   →  To print results
```

(b.) Routine --- DIST

Entry:	p	AB	XY(DIST)
		50	p
	p + 1	26	(DIST)
		ab	xyF

Variations and Explanations of Parameters:

	AB:	51 -- Carry-over to "Front" panel	
		5S -- Carry-over to "Back" panel	
	XY:	21 -- Distribute + Carry-over	
		66 -- Carry-over only	
ab	xy:	00 13 -- Flexural C.O. or Shear C.O.	
		33 14 -- Flexure-shear C.O. or Shear-flexural C.O.	

Loc.	Orders		Notes		
	(DIST)				
0	S5	F			Plant links and switches
	42	4L			
1	46	141L			
	46	165L			
2	L4	(1)			
	42	6L			
3	42	67L			
	L4	(1)	Plant link		
4	42	104L			
	L5	F	by 0'	Plant C.O. to "Front" or "Back" panel switch	
5	10	12F			
	L4	104L			
6	46	66L			
	L5	F	by 2'		
7	42	9L			
	42	15L			
8	L5	15F	Plant "Direct" or Reverse" switch		
	42	20L			
9	50	F	waste		
	L5	F	by 7		
10	46	121L			
	46	142L			

```
11    L4  (01)
      46  127L
12    46  148L        Plant (1F) and (2F)
      42  136L          or  (3F) and (4F)
13    42  154L
      L4  (1)
14    42  137L
      42  157L
15    50  F      waste
      L5  F      by 7'
16    L0  (3FF)
      36  19L
17    L5  45L
      46  140L
18    46  163L        Plant C.O. parameters
      22  20L         for Mom. or Reac.
19    L5  139L
      46  140L
20    46  163L
      26  F      by 8'  --> To "Direct" or Reverse"
21    L5  76L
      46  (DCOEF)      Initialize Distribution Factors
22    L5  88L
      46  (SCOEF)
23    50  0F
      75  (0.01)
24    66  540F
      S5  F
25    40  0F           Deformation of Joint relaxed
      50  1F
26    75  (0.01)
      66  540F
27    S5  F
      40  1F
28    41  167L  — clear C.O. counter
      L5  (DCOEF)
```

```
29    46  33L
      46  36L
30    L5  (D101)
      46  168L
31    46  35L
      L5  (D201)
32    46  37L
      46  38L
33    51  F
      75  OF
34    00  1F
      26  168L
35    40  F         by 31
      51  1F
36    75  F         by 29'
      00  1F
37    L4  F         by 32
      50  F         waste
38    40  F         by 32'
      L5  (D101)
39    L4  (01)
      50  F         parameter planted by
40    40  (D101)
      L5  (D201)
41    L4  (01)
      50  F         parameter planted by
42    40  (D201)
      L5  (DCOEF)
```

Deformation of edge relaxed

```
43    L4  (01)
      40  (DCOEF)
44    F5  167L
      40  167L
45    50  147L              End test
      LO  (PM)
46    36  47L
      22  28L       →       Loop
47    41  167L          clear C.O. counter
      L5  (SCOEF)
48    46  53L
      46  56L
49    L5  (S101)
      46  54L
50    46  55L
      L5  (S201)
51    46  57L
      46  58L
52    50  F             waste
      51  OF
53    75  F             by 48
      00  1F
54    L4  F             by 49'
      50  F             waste
55    40  F             by 50
      51  1F
56    71  F             by 48'
      00  1F
57    L4  F             by 51                      Forces of edge relaxed
      50  F             waste
58    40  F             by 51'
      L5  (S101)
59    L4  (01)          parameter planted
      50  F             by
60    40  (S101)
      L5  (S201)
```

```
61    L4  (01)       parameter planted
      50  F          by
62    40  (S201)
      L5  (SCOEF)
63    L4  (01)
      40  (SCOEF)
64    F5  167L
      40  167L
65    50  F          End test
      L0  (PM)
66    32  (Link)F    by 6; → to C.O. to
                        Front or Back
      22  47L  → Loop
67    41  167L       clear C.O. counter
      L5  (Link)F    by 3
68    00  8F                     Initialize (FSCO)
      L4  100L
69    46  (FSCO)
      L5  (F102)
70    46  73L
      46  74L
71    L5  (FSCO)
      46  72L
72    51  F          by 71'   parameter
      7J  0F                  planted by
73    L4  F          by 70
      50  25F
74    40  F          by 70'                        C.O. to Near Adj. edge
      L5  (F102)
75    L4  (01)       parameter
      50  F          planted by
76    50  541F       address of K_1 or T_1
      40  (F102)
77    L5  (FSCO)
      L4  (01)
```

78	40	(FSCO)		
	F5	167L		
79	40	167L	End test	
	L0	(PM)		
80	36	81L		
	22	69L	→ Loop	
81	41	167L	clear C.O. counter	
	L5	(F103)		
82	46	85L		
	46	86L		
83	L5	(FSCO)		
	46	84L		
84	51	F	by 83'	
	7J	0F		
85	L4	F	by 82	
	50	30F		
86	40	F	by 82'	
	L5	(F103)		
87	L0	(01)		C.O. to Opposite edge
	50	F		
88	50	558F	address of Q_{11} of Q'_{11}	
	40	(F103)		
89	L5	(FSCO)		
	L4	(01)		
90	40	(FSCO)		
	F5	167L		
91	40	167L	End test	
	L0	(PM)		
92	36	93L		
	22	81L	→ Loop	
93	41	167L	clear C.O. counter	
	L5	(F104)		
94	46	97L		
	46	98L		
95	L5	(FSCO)		
	46	96L		

```
96     51  F          by 95'      parameter
       7J  OF                     planted by
97     L4  F          by 94
       50  121L
98     40  F          by 94'
       L5  (F104)
99     L0  (01)       parameter
       50  F          planted by
100    50  63F        address for k11 or t11
       40  (F104)
101    L5  (FSCO)
       L4  (01)
102    40  (FSCO)
       F5  167L
103    40  167L       End test
       L0  (PM)
104    50  24L        waste
       36  (Link)F    by 4; → Out to "Cont"
105    22  93L    → Loop
       L5  (L4)
106    40  75L
       L5  (L0)
107    40  99L
       L5  F          by 139' or 163'
108    40  72L
       40  96L
109    L5  73L
       42  69L
110    42  74L
       42  76L
111    L4  (1)
       42  81L
112    42  86L
       42  88L
113    L4  (1)
       42  93L
```

C. O. to Far Adj. edge

Initialize for
C.O. to "Front" panels

```
114    42  98L
       42  100L
115    26  67L   --> To Obey C.O.
       L5  (L0)
116    40  75L
       L5  (L4)
117    40  99L
       L5  F         by 140' or 164'
118    40  72L                                    Initialize for C.O.
       40  96L                                    to "Back panels
119    L5  1F
       40  0F
120    L5  85L
       22  109L  --> To Obey C.O. via 109'
121    L5  (1F)      by 10
       46  (D101)
122    L4  (M-1)
       L4  (M-1)
123    40  (F102)
       L4  (N-1)
124    L4  (M-1)
       L0  (01)
125    40  (F103)
       L4  (N-1)
126    40  (F104)
       50  84L       address of 7J par.
127    L5  (2F)      by 11'
       46  (D201)
128    L4  (M-1)
       L4  (M-1)
129    L4  (N-1)
       L0  (01)
130    40  (F202)
       L4  (M-1)
131    40  (F203)                                 Initialize for
       L4  (01)                                   "Direct" distribution
```

```
132     40  (F204)
        L5  (L4)
133     40  39L
        40  41L
134     40  59L      Set (L4) and (L0)
        40  61L      parameters
135     L5  (L0)
        40  87L
136     L5  (M-1)
        L4  (3F)     by 12'
137     40  (S101)
        L5  (4F)     by 14
138     L4  (M-1)
        40  (S201)
139     L5  126L
        42  107L     plant 7J and 79
140     L5  147L     parameters
        42  117L
141     26  (Link)F  by 1; → To "Dist + C.O. or
        50  142L     address    "C.O. only"
                     of SL
142     L5  (1F)     by 10'
        L4  (M-1)
143     L0  (01)
        46  (D101)
144     L4  (M-1)
        L4  (N-1)
145     40  (F104)
        L4  (01)
146     40  (F103)
        L4  (M-1)
147     40  (F102)
        50  166L     waste
148     L5  (2F)     by 12
        L4  (M-1)
149     L0  (01)
        46  (D201)
```

```
150     L4  (M-1)
        L4  (01)
151     40  (F204)
        L4  (N-1)
152     40  (F203)
        L4  (M-1)
153     L4  (N-1)
        L0  (01)
154     40  (F202)
        L5  (3F)       by 13                     Initialize for
155     L4  (M-1)                                "Reverse" distribution
        L4  (M-1)
156     L0  (01)
        40  (S101)
157     L5  (M-1)
        L4  (4F)       by 14'
158     L4  (M-1)
        L0  (01)
159     40  (S201)
        L5  (L0)
160     40  39L
        40  41L
161     40  59L        Set (L4) and (L0)
        40  61L        parameters
162     L5  (L4)
        40  87L
163     L5  147L
        42  107L       Plant 7J and 79
164     L5  126L       parameters
        42  117L
165     26  (Link)F    by 1'; --> To "Dist + C.O."
        00  F          waste      or "C.O. only"
166     51  F
        79  0F         79 parameter
167     00  F
        00  F          C.O. Counter
168     L4  F          by 30'
        26  35L
```

(c.) Routine --- HV

Entry:	p	50	(REF) or (AUX)
		50	p
	p + 1	26	(HV)
		50	(AUX) or (REF)

Loc.	Orders		Notes
	(HV)		
0	K5	F	Plant Link
	42	2L	
1	46	25L	
	L4	(1)	
2	42	27L	
	L5	F	by 0'
3	42	26L	
	F5	7F	Compute Reference Joint for E-W edge
4	F4	7F	
	40	OF	
5	50	OF	
	75	6F	
6	75	(M)	
	S5	F	
7	40	(MAX)	
	L5	(R)	
8	L4	(R)	
	LO	(1)	
9	40	OF	
	50	OF	
10	75	7F	
	75	(N)	
11	S5	F	
	L4	(MAX)	
12	40	OF	
	L5	(S)	
13	LO	(1)	
	40	1F	

```
14    50  1F
      75  (N)
15    K5  F
      L4  0F
16    40  1F
      L5  (S)
17    L4  (S)
      L0  (1)
18    40  0F
      50  0F
19    75  6F
      75  (M)
20    S5  F                    Compute Reference
      40  0F                   Joint for N-S edge
21    L5  (R)
      L0  (1)
22    40  2F
      50  2F
23    75  (M)
      K5  F
24    L4  0F
      50  F       waste
25    42  F
      50  F       waste
26    L5  1F
      42  F
27    50  F       waste
      26  (Link)F
```

(d.) Routine --- PR1

Entry: p 50 (RE) or (PLAY)

50 p

p + 1 26 (PR1)

Loc.	Orders	Notes
	(PR1)	
0	K5 F	Plant Link and Initialize
	42 35L	
1	46 14L	
	46 20L	
2	46 30L	
	46 33L	
3	L5 14L	
	42 13L	
4	L5 20L	
	42 19L	
5	L5 (REF)	Test for Upper edge
	L0 (MAX)	
6	32 8L	
	L5 (S)	
7	F0 7F	
	36 27L	
8	26 10L	
	L5 (R)	
9	F0 6F	
	36 27L	
10	L5 (REF)	
	L4 (MOM)	
11	40 (HDRUM)	
	L5 (AUX)	
12	L4 (MOM)	
	40 (VDRUM)	
13	50 F	
	26 F	by 3' or 27'

14	51	F	
	50	14L	Play-Re: Rotations and Moments for "Front" panel
15	26	(PR)	
	50	(1F)	
16	L5	(REF)	
	L4	(SHEAR)	
17	40	(HDRUM)	
	L5	(AUX)	
18	L4	(SHEAR)	
	40	(VDRUM)	
19	50	F	
	26	F	by 4' or 28'
20	51	F	
	50	20L	Play-Re: Deflections and Reactions for "Front" panel
21	26	(PR)	
	50	(3F)	
22	L5	(REF)	
	L0	(MAX)	
23	32	25L	
	L1	(S)	
24	L4	(1)	
	32	35L	Test for Lower edge
25	26	27L	
	L1	(R)	
26	L4	(1)	
	32	35L	
27	L5	30L	
	42	13L	
28	L5	33L	
	42	19L	
29	26	10L	
	50	F	waste
30	5S	F	
	50	30L	Play-Re: Rotations and Moments for "Back" panel
31	26	(PR)	
	50	(1F)	

```
32    26  16L
      50  F       waste
33    5S  F
      50  33L         Play-Re:
34    26  (PR)        Deflections and Reactions
                      for "Back" panel
      50  (3F)
35    50  F       waste
      22  (Link)F
```

(e.) Routine --- PR

Entry	p	51 or 5S	(RE) or (PLAY)	
		50	p	
	p + 1	26	(PR)	
		50	(1F) or (3F)	

Loc.	Orders			Notes
	(PR)			
0	K5	F		Plant Link and Initialize, and plant parameters
	42	64L		
1	42	3L		
	46	7L		
2	01	27F		
	L4	30L		
3	46	21L		
	L5	F	by 1	
4	42	30L		
	42	39L		
5	L4	(1)		
	42	28L		
6	42	41L		
	50	F	waste	
7	L5	F	by 1'	
	50	F	waste	
8	40	47L		
	40	50L		
9	40	53L		
	40	56L		
10	40	59L		
	40	62L		
11	L5	46L		
	42	47L		
12	L5	49L		
	42	50L		
13	L5	52L		
	42	53L		

```
14      L5  55L
        42  56L
15      L5  58L
        42  59L
16      L5  61L
        42  62L
17      L5  (M-1)
        46  49L
18      46  52L
        46  58L
19      46  64L
        L5  (N-1)
20      46  55L
        46  61L
21      22  F           by 3
        L5  (HDRUM)
22      40  48L
        L4  (DIFF)
23      40  51L
        L4  (HM)
24      40  57L
        L0  (HM)
25      L0  (HM)
        40  63L
26      L5  (VDRUM)          Initialize drum addresses
        L4  (DIFF)           for "Front" panels
27      40  60L
        L4  (VN)
28      40  54L
        L5  (2F)
29      L4  (M-1)
        46  62L
30      50  964L        waste
        L5  (1F)
31      26  42L
        L5  (HDRUM)
32      L0  (HM)
        40  48L
```

```
33      L4  (DIFF)
        40  51L
34      L0  (HM)
        40  57L
35      L4  (HM)
        L4  (HM)              Initialize drum addresses
36      40  63L               for "Back" panels
        L5  (VDRUM)
37      L4  (DIFF)
        F0  (PM)
38      40  60L
        L4  (VN)
39      40  54L
        L5  (1F)
40      L4  (M-1)
        46  62L
41      50  F       waste
        L5  (2F)
42      46  47L
        L4  (M-1)
43      46  50L
        L4  (M-1)             Initialize force and
44      46  53L               deformation "Boxes"
        L4  (N-1)
45      46  56L
        L4  (M-1)
46      46  59L
        50  47L     waste
47      00  F
        00  F
48      26  (Y1)    Play-Re:
        00  F       Deformation "Boxes"
49      00  F
        50  50L     waste
50      00  F
        00  F
51      26  (Y1)
        00  F
```

```
52    00  F
      50  53L        waste
53    00  F
      00  F
54    26  (Y1)
      00  F
55    00  F
      50  56L
56    00  F
      00  F
57    26  (Y1)                         Play-Re:
      00  F                            Force "Boxes"
58    00  F
      50  59L
59    00  F
      00  F
60    26  (Y1)
      00  F
61    00  F
      50  62L
62    00  F
      00  F
63    26  (Y1)
      00  F
64    00  F
      22  (Link)F  → Out to Link
```

(f.) Constants, Counters, and Parameters

Loc.	Sym. Add.	Orders		Notes
9	(1F)	00	680F	Initial Addresses of deformation "Boxes"
		00	F	
10	(2F)	00	765F	
		00	F	
11	(3F)	00	850F	
		00	F	
12	(4F)	00	935F	
		00	F	
13		00	F	Parameters
		00	F	
14		00	F	
		00	F	
15		00	F	Switch parameter
		00	F	
16	(2-1)	00	xF	x = (2F) - (1F)
		00	F	
17	(END)	00	F	
		00	xF	x = 2[h(m-1)(v+1) + v(n-1)(h+1)]
18	(111)	00	F	
		00	xF	x = h(v+1), number of E-W reference corners
19	(222)	00	F	
		00	F	x = (111) + v(h+1), number of E-W and N-S reference corners
20	(M)	00	F	Constants
		00	1S4	
21	(N)	00	F	
		00	1S5	
22	(DIFF)	00	F	Difference in drum locations between corresponding force and deformation "Boxes"
		00	1000F	
23	(D101)	00	F	Address of "Front" panel temporary deformation "Box".
		00	F	
24	(F101)	00	F	
		00	F	

25	(F102)	00	F	
		00	F	
26	(F103)	00	F	Addresses of "Front" panel temporary force "Boxes"
		00	F	
27	(F104)	00	F	
		00	F	
28	(D201)	00	F	Address of "Back" panel temporary deformation "Box"
		00	F	
29	(F201)	00	F	
		00	F	
30	(F202)	00	F	Addresses of "Back" panel temporary force "Boxes"
		00	F	
31	(F203)	00	F	
		00	F	
32	(F204)	00	F	
		00	F	
33	(S101)	00	F	
		00	F	Addresses of "Front" and Back" panels temporary auxiliary force "Boxes"
34	(S201)	00	F	
		00	F	
35	(DCOEF)	00	F	Address of K_i or T_i
		00	F	
36	(SCOEF)	00	F	Address of q_{ji} or q'_{ji} for edge
		00	F	released
37	(FSCO)	00	F	Address of $k_{ji} + q_{ji}$ or $t_{ji} + q'_{ji}$
		00	F	
38	(+)	26	(Y1)	
		00	9000F	Drum loc. of K_1
39	(T)	26	(Y1)	
		00	10300F	Drum loc. of T_1
40	(C)	00	F	Edge joint counter
		00	F	
41	(C1)	00	F	Counter for start of "Reverse"
		00	F	
42	(C2)	00	F	Counter to determine joint relaxed
		00	F	
43	(S)	00	F	Number of the E-W edge
		00	F	

44	(R)	00	F	Number of the N-S edge
		00	F	
45	(HM)	00	F	Difference in Drum Loc. between corresponding joints on two successive E-W edges
		00	F	
46	(VN)	00	F	Difference in Drum loc. between corresponding joints on two successive N-S edges
		00	F	
47	(REF)	00	F	Reference points
		00	F	
48	(AUX)	00	F	
		00	F	
49	(MOM)	26	(Y1)	Correspondence constants
		00	3000F	
50	(SHEAR)	26	(Y1)	
		00	5000F	
51	(HDRUM)	00	F	For initializing force and deformation "Boxes"
		00	F	
52	(VDRUM)	00	F	
		00	F	
53	(PM)	00	F	Parameters
		00	F	
54	(M-1)	00	F	
		00	F	
55	(N-1)	00	F	
		00	F	
56	(MAX)	00	F	Total number of edge joints
		00	F	
57	(1FF)	00	(1F)	Parameters
		00	(3F)	
58	(3FF)	00	(3F)	
		00	(1F)	
59	(PLAY)	50	F	Playback parameter
		50	F	
60	(RE)	JO	F	Record parameter
		50	F	
61	(MV)	00	F	Moment or Reaction counter: 0 = Mom.; 1 = Reac.
		00	F	

```
62      (EC)      00  F                    Counter for the number of joints
                                           that have attained convergence
                  00  F
63      (HVC)     00  F
                                           E-W or N-S edge relaxed counter
                  00  F
64      (1)       00  F
                  00  1F
65      (01)      00  1F                     Constants
                  00  F
66      (0.01)    00  F
                  00  0100 0000 0000J
67      (L4)      L4  (01)
                  50  F
68      (L0)      L0  (01)             Parameters
                  50  F
          00  109K
 0        L5  (2F)
          L0  (1F)
 1        40  (2-1)
          F5  7F
 2        40  0F
          51  0F
 3        75  6F
          S5  F
 4        40  (111)
          F5  6F
 5        40  0F
          51  0F
 6        75  7F
          S5  F
 7        40  1F
          L4  (111)             An interlude to compute some constants
 8        40  (222)
          51  (111)
 9        75  4F
          S5  F
10        40  0F
          51  1F
```

11	75	5F
	S5	OL
12	40	1F
	L4	1F
13	L4	OF
	L4	OF
14	40	(END)
	50	11L
15	26	999F
	50	F
16	26	OL
	26	1N

B-2. <u>Code SM-19</u>

Title --- Determination of $w_p + w_c$

Type --- Complete program

Number of Words --- 273

Duration --- (mxn) $\left[\frac{1}{24} + \frac{pN}{1500}\right]$ min., for p-digit output where N = No. of panels.

Pre-set Parameters:

S3 --- Location of first element of matrix $\underline{\alpha}_j$ or $\underline{\beta}_j$.

S4 --- Location of first element of matrix $\underline{w}_c + \underline{w}_p$

S5 --- Total number of elements in one matrix

Sub-routines:

$\left.\begin{array}{l}\text{N12 --- Infraput}\\ \text{P16 --- Infraprint}\\ \text{Y1 --- Drum transfer}\end{array}\right\}$ ILLIAC Library

William's Memory Assignments:

0, 1, 2 --- Temporary storage

3, 4, 5 --- Pre-set parameters

6 -- 45 --- Y1

46 - 203 --- SM-19 (Main program)

300 - 355 --- P16 (during output)

400 - 462 --- Input routine (during input)

204 - 276 --- Edge rotations, θ_j, or edge deflections, Δ_j, of one panel

277 - 637 --- Temporary storage for one matrix $\underline{\alpha}_j$ or $\underline{\beta}_j$

638 - 998 --- Partial sum $\underline{w}_p + \underline{w}_c$

Description:

This program computes the deflections, w_c, of all interior points due to the edge rotations and edge deflections and form the sum $w_p + w_c$ for all interior points. This involves the addition of the following matrices:

$$\sum_{j=1}^{2(m+n)} \underline{\alpha}_j \cdot \theta_j + \underline{\beta}_j \cdot \Delta_j + \underline{w}_p = \underline{w}$$

where: $\underline{w}$ = total deflection matrix

$\underline{w}_p$ = w_p matrix

$\underline{\alpha}_j$ = deflection matrix of the interior points due to $\theta_j = 1$

$j = 1, 2, \ldots . 2(m + n)$

$\underline{\beta}_j$ = deflection matrix of the interior points due to $\Delta_j = 1$

θ_j = edge rotation at j, a scalar

Δ_j = edge deflection at j, a scalar

It is necessary only to determine $\underline{\alpha}_j$ and $\underline{\beta}_j$ for $j = 1, 2, \ldots . m/2$ (or $\frac{m+1}{2}$ for m odd). In order to determine the interior point, I, correctly during the calculation of the partial sum $w_p + w_c$, the following formulas are required:

For j in the range;

$0 < j \leq \frac{m}{2}$ (or $\frac{m+1}{2}$); $\quad I = (y - 1)m + x$

$\frac{m}{2} < j \leq m$; $\quad I = (ym + 1) - x$

$m < j \leq m + \frac{n}{2}$(or $m + \frac{n+1}{2}$); $\quad I = m(n - x) + y$

$m + \frac{n}{2} < j \leq m + n$; $\quad I = (x - 1)m + y$

$m + n < j \leq m + n + \frac{m}{2}$ (or $m + n + \frac{m+1}{2}$); $\quad I = m(n + 1 - y) - (x - 1)$

$(m + n + \frac{m}{2}) < j \leq 2m + n$: $\quad I = (n - y)m + x$

$2m + n < j \leq 2m + n + \frac{n}{2}$ (or $2m + n + \frac{n+1}{2}$); $\quad I = xm - y + 1$

$2m + n + \frac{n}{2} < j \leq 2(m + n)$; $\quad I = m(n - x + 1) - (y - 1)$

where; x = the number of the column of the matrix

y = the number of the row of the matrix

in which the column number and row number are taken with respect to the edge where j is.

Form of Output:

The deflections of all interior points in a panel are printed to a specified number of digits (up to 12). These are printed in a single column starting with the upper-most row and printed from left to right. Each panel is printed completely in succession in the following order; panels (1,1), (1,2), ... (2,1), (2,2), ... (3,1), (3,2)....... The deflections of each panel is terminated with the character N.

Preparation of Data Tapes:

The initial deflections of a panel, w_p, are typed and ended with the character N. Zeros are represented with a "+". These initial deflections are followed by the edge rotations and edge deflections of the panel ending each edge deformations with an N.

All these information, in the given order, should be done for all the panels in which the total deflections, w, are desired.

Routine --- SM-19

Loc.	Orders	Notes	
	(SM)		
0	L5 (I)		
	42 4L		
1	42 5L		
	L5 (SET)	Plant addresses	
2	42 3L		
	L5 (D101)		
3	46 4L		
	50 F	by 2	
4	7J F	by 3	
	L4 F	by 0'	Computes partial sum $w_p + w_c$
5	50 34L	waste	
	40 F	by 1	
6	F5 (SET)		
	40 (SET)		
7	F5 (P)		
	40 (P)		
8	F0 (M)		
	32 9L		
9	26 13L	→ To compute I via 13	
	F5 (J)		
10	40 (J)	Advance row number	
	L5 (1)		
11	40 (P)		
	L5 (J)		
12	F0 (N)	End test of one θ_j or Δ_j	
	32 13L		
13	26 F	— Switch	
	L5 (D101)		
14	L4 (01)	Parameter planted by 28', 39', etc.	
	50 F		
15	40 (D101)		
	F5 27L		

16	40	27L	End test of one range of j	
	L0	(M/2)		
17	50	F	waste	
	36	F	— Switch	
18	50	S3		
	50	18L		
19	26	(Y1)	Planted by 26	Play-back one $\underline{\alpha_j}$ or $\underline{\beta_j}$
	00	3000F		
20	00	S5		
	L5	19L		
21	F4	5F		
	40	19L		
22	L5	3F		
	40	(SET)		
23	L5	(1)	Initialize column and row number of matrix	
	40	(P)		
24	40	(J)		
	50	F		
25	26	F	— Switch	
	L5	(DC11)	by 126' or 128'	
26	40	19L		
	26	18L		
27	00	F	Counter	
	00	F		
28	L5	(L4)		
	40	14L		
29	L5	5L		
	46	13L		
30	46	25L		
	L5	33L		
31	42	17L		
	L5	(D1F)		
32	40	(D101)		
	41	27L	Clear counter	
33	22	25L		
	50	39L	waste	

```
34   L5  (J)
     L0  (1)
35   40  0F
     50  0F
36   75  (M)       Compute I for the range 0 < j ≤ m/2
     S5  F
37   L4  (P)
     L4  4F
38   40  (I)
     26  0L
39   L5  (L0)
     40  14L
40   L5  47L
     46  13L
41   46  25L
     L5  45L
42   42  17L
     L5  (D1F)
43   L4  (M1)
     40  (D101)
44   L5  (1)       Set counter = 1
     40  27L
45   22  25L
     50  50L       waste
46   50  (J)
     75  (M)
47   K5  46L       Compute I for the range m/2 < j ≤ m
     L0  (P)
48   L4  4F
     40  (I)
49   50  56L       waste
     26  0L
50   L5  (L4)
     40  14L
51   L5  49L
     46  13L
```

```
52   46  25L
     L5  58L
53   42  17L
     L5  (D2F)
54   40  (D101)
     50  F          waste
55   41  27L        Clear counter
     22  25L
56   L5  (N)
     L0  (P)
57   40  0F
     50  0F
58   75  (M)        Compute I for the range m < j ≤ (n + 1)/2
     S5  61L
59   L4  (J)
     L4  4F
60   40  (I)
     26  0L
61   L5  (L0)
     40  14L
62   L5  67L
     46  13L
63   46  25L
     L5  70L
64   42  17L
     L5  (D2F)
65   L4  (M1)
     40  (D101)
66   L5  (1)        Set counter = 1
     40  27L
67   50  68L        waste
     22  25L
68   L5  (P)
     L0  (1)
69   40  0F
     50  0F
```

```
70    75  (M)      Compute I for the range m + n/2 < j ≤ m + n
      S5  73L
71    L4  (J)
      L4  4F
72    40  (I)
      26  0L
73    L5  (L4)
      40  14L
74    L5  78L
      46  13L
75    46  25L
      L5  81L
76    42  17L
      L5  (D3F)
77    40  (D101)
      41  27L      Clear counter
78    50  79L      waste
      22  25L
79    F5  (N)
      L0  (J)
80    40  0F
      50  0F
81    75  (M)      Compute I for the range m + n < j ≤ m + n + m/2
      K5  84L
82    L0  (P)
      L4  4F
83    40  (I)
      26  0L
84    L5  (L0)
      40  14L
85    L5  90L
      46  13L
86    46  25L
      L5  93L
87    42  17L
      L5  (D3F)
```

```
88    L4  (M1)
      40  (D101)
89    L5  (1)       } Set counter = 1
      40  27L
90    50  91L         waste
      22  25L
91    L5  (N)
      L0  (J)
92    40  OF
      50  OF
93    75  (M)       } Compute I for the range m + n + m/2 < j ≤ 2m + n
      S5  96L
94    L4  (P)
      L4  4F
95    40  (I)
      26  OL
96    L5  (L4)
      40  14L
97    L5  103L
      46  13L
98    46  25L
      L5  101L
99    42  17L
      L5  (D4F)
100   40  (D101)
      41  27L         Clear counter
101   22  25L
      50  106L        waste
102   50  (P)
      75  (M)
103   K5  102L      } Compute I for the range 2m + n < j ≤ 2m + n + n/2
      L0  (J)
104   L4  4F
      40  (I)
105   26  OL
      50  (DC11)      waste
```

```
106   L5  (LO)
      40  14L
107   L5  112L
      46  13L
108   46  25L
      L5  115L
109   42  17L
      L5  (D4F)
110   L4  (M1)
      40  (D101)
111   L5  (1)
      40  27L
112   50  113L        waste
      22  25L
113   F5  (N)         ⎫
      LO  (P)         ⎪
114   40  0F          ⎪
      50  0F          ⎪
115   75  (M)         ⎬ Compute I for the range 2m + n + n/2 < j ≤ 2(m + n)
      K5  118L        ⎪
116   IO  (J)         ⎪
      L4  4F          ⎪
117   40  (I)         ⎭
      26  0L
118   50  300F        ⎫
      50  118L        ⎪
119   26  (Y1)        ⎬ Play-back (P16) into 300 ....
      00  2600F       ⎪
120   00  56F         ⎭
      92  143F        4 CRLF
121   50  400F        ⎫
      50  121L        ⎪
122   26  (Y1)        ⎬ Play-back Input routine into 400 .....
      00  2700F       ⎪
123   00  44F         ⎭
      L5  137L        ⎫
124   LO  (1)         ⎬ Test for θ_j or Δ_j computed
      32  127L        ⎭
```

125	F5 137L		
	40 137L		
126	L5 131L		Plant (DC22)
	42 25L		
127	26 442F		→ To read in δ_i
	41 137L		Clear $\theta\Delta$ counter
128	L5 105L		Plant (DC11)
	42 25L		
129	F5 4F		
	42 130L		
130	50 (1)		Clear q_1
	L5 F		by 129'
131	00 1F		
	50 (DC22)		
132	50 nF		Prints to n digits
	50 132L		
133	26 (P16)		
	92 131F		CRLF
134	F5 130L		
	40 130L		
135	L0 (END)		
	36 130L		→ Loop
136	92 770F		N
	24 439F		→ To read in $w_p + \varphi_i$
137	00 F		$\theta\Delta$ counter
	00 F		
138	(P)	00 F	Parameters
		00 F	
139	(J)	00 F	
		00 F	
140	(I)	00 F	
		00 F	
141	(SET)	00 F	
		00 F	
142	(D101)	00 F	
		00 F	

143	(M)	00	F	
		00	cF	c = number of columns in matrix
144	(N)	00	F	
		00	rF	r = number of rows in matrix
145	(M/2)	00	F	
		00	xF	x = No. of θ_j or Δ_j in $0 < j \leq \frac{m}{2}$
146	(DC11)	26	(Y1)	Drum loc. of $\alpha_1(1,1)$
		00	3000F	
147	(DC22)	26	(Y1)	Drum loc. of $\beta_1(1,1)$
		00	6500F	
148	(M1)	00	(m-1)F	No. of θ_j for one edge minus 1
		00	F	
149	(1)	00	F	Constants
		00	1F	
150	(01)	00	1F	
		00	F	
151	(L4)	L4	(01)	Parameters
		50	F	
152	(L0)	L0	(01)	
		50	F	
153	(D1F)	00	205F	Address of θ_1 or Δ_1 in $0 < j \leq m$
		00	F	
154	(D2F)	00	222F	Address of θ_1 or Δ_1 in $m < j \leq m + n$
		00	F	
155	(D3F)	00	239F	Address of θ_1 or Δ_1 in
		00	F	$m + n < j \leq 2m + n$
156	(D4F)	00	256F	Address of θ_1 or Δ_1 in
		00	F	$2m + n < j \leq 2(m + n)$
157	(END)	J0	F	End constant
		L5	(c x r)1S4	

```
(P16)   00  300K
(N12)   00  400K

        00  K
  0     50  1S4       Input w_p of one panel
        50  0L
  1     26  (N12)
        26  3L
  2     00  F         Counter
        00  F
  3     50  205F      Input θ_j or Δ_j of one panel
        50  3L
  4     26  (N12)
        26  28(SM)  → To Routine
  5     50  1S4
        50  5L
  6     26  (N12)
        50  F      waste
  7     J0  1S4
        50  7L
  8     26  (Y1)
        00  3000F
  9     00  S5
        L5  8L
 10     F4  5F
        40  8L
 11     F5  2L              Overwritten  after
        40  2L              input of α_j and β_j
 12     L0  (M/2)
        34  14L
 13     26  5L
        50  0L     waste
 14     41  2L
        L5  13L
 15     42  12L
        L5  (DC22)
```

Input routine

16	40	8L
	26	5L
17	JO	300F
	50	17L
18	26	(Y1)
	00	2600F
19	00	56F
	50	F
20	JO	400F
	50	20L
21	26	(Y1)
	00	2700F
22	00	44F
	26	5L
23	26	17L
	24	1N